Optimization over the intersection of manifolds

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Outline

Background: optimization with geometric constraints

Riemannian optimization problem

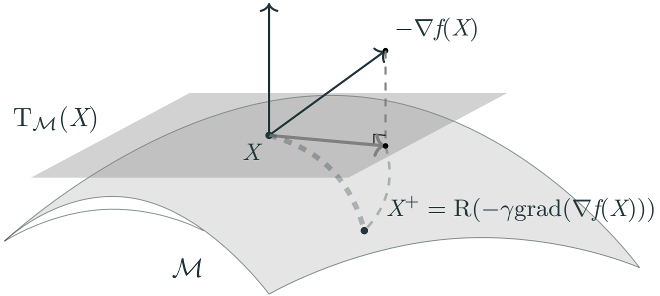
$$\boxed{\min_{X \in \mathcal{M}} f(X)}$$

with $\mathcal{M} \subseteq \mathbb{R}^m$ as a smooth manifold

Geometric ingredients

- Tangent space $\mathrm{T}_{\mathcal{M}}(X)$
- Riemannian gradient $\mathrm{grad}_{\mathcal{M}} f(X)$
- Retraction $\mathrm{R}_x : \mathrm{T}_{\mathcal{M}}(X) \to \mathcal{M}$

Riemannian gradient descent



Specific manifolds: theory and applications

Stiefel and Grassmann manifolds $\{X \in \mathbb{R}^{m\times p} \mid X^\top X = I_p\}$

- Principal component analysis problems [Jolliffe-Trendafilov-Uddin'03; Xu-Jiang-Liu-So'26]
- Computer vision and statistics on manifolds [Turaga-Veeraraghavan-Chellappa'08]
- Electronic structure calculations [Liu-Wang-Wen-Yuan'13; Jiang-Dai'16; Gao-Liu-Yuan'18]

Fixed-rank matrix manifold $\{X \in \mathbb{R}^{m\times n} \mid \mathrm{rank}(X) = r\}$

- Low-rank matrix completion [Vandereycken'13]
- Magnetic resonance imaging (MRI) [Banco-Aeron-Hoge'16; Choi-Bao-Zhang'18; Fessler'20]
- Data analysis, e.g., weather forecast [Loucheur-Absil-Journée'23] and exoplanet detection [Daglayan-Vary-Absil-Cantalloube-Christiaens-Gillis-Jacques-Leplat-Absil'24]

Fixed-rank tensor manifolds $\{\mathcal{X} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d} \mid \mathrm{rank}(\mathcal{X}) = \mathbf{r}\}$

- Tucker: tensor completion and parametric PDEs [Kressner–Steinlechner–Vandereycken'14]
- Tensor train: high-dimensional completion [Steinlechner'16; Peng-Zhu-Gao-Wang-Yuan'25]

Additional structures imposed on the manifold

$$\begin{aligned} \min_{X \in \mathcal{E}} \quad & f(X) \\ \text{s.t.} \quad & h(X) = 0 \\ & X \in \mathcal{M} \end{aligned}$$

Blanket assumptions

- Euclidean space $\mathcal{E}$ accommodates $\mathbb{R}^m$, $\mathbb{R}^{m \times n}$, or $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$
- Objective $f \colon \mathcal{E} \to \mathbb{R}$ is continuously differentiable
- Differential $\mathrm{D}h$ has **full rank** in a neighborhood of

$$\mathcal{H} := \{X \in \mathcal{E} \mid h(X) = 0\}$$

Manifold + additional constraints

Additional structures imposed on the manifold

$$\begin{aligned} \min_{X \in \mathcal{E}} \quad & f(X) \\ \text{s. t.} \quad & h(X) = 0 \\ & X \in \mathcal{M} \end{aligned}$$

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$$\mathcal{H} := \{X \in \mathcal{E} \mid h(X) = 0\}$$

$\mathcal{H}$ is a **smooth manifold** embedded in $\mathcal{E}$

Application I. Low rank + additional constraints

Fixed-rank constraint + an additional constraint

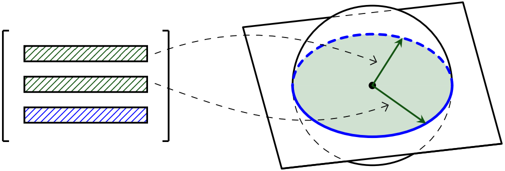
$$
\begin{aligned}
\min_{X \in \mathbb{R}^{m\times n}} \quad & f(X) \\
\text{s. t.} \quad & h(X) = 0 \\
& X \in \mathcal{M}_r = \{X \in \mathbb{R}^{m\times n} \mid \operatorname{rank}(X) = r\}
\end{aligned}
$$

Low rank + oblique manifold

Low-rank data fitting on sphere [Chu-Del Buono-Lopez-Politi'05 SIMAX]

$$\mathcal{H} = \mathrm{Ob}(m,n) := \{X \in \mathbb{R}^{m\times n} \mid \operatorname{diag}\left(XX^\top\right) - \mathbf{1} = \mathbf{0}\}$$

$\mathbb{R}_2^{3\times 4} \cap \mathrm{Ob}(3,4) \ni$



Low rank + Frobenius sphere

Approximation of graph similarity matrices [Cason-Absil-Van Dooren'13 LAA]

$$\mathcal{H} = \mathrm{S}_{\mathrm{F}}(m, n) := \{X \in \mathbb{R}^{m \times n} \mid \|X\|_{\mathrm{F}}^2 - 1 = 0\}$$

Low rank + hyperbolic manifold

Approximation of hyperbolic embeddings [Jawanpuria-Meghwanshi-Mishra'19 CDC]

$$\mathcal{H} = \mathbb{H}_n^m := \{X \in \mathbb{R}^{m \times n} \mid \mathrm{diag}(X \mathrm{Diag}(-1, 1, \ldots, 1) X^\top) - \mathbf{1} = \mathbf{0}\}$$

Low rank + Frobenius sphere (TT tensors)

Applications in quantum information theory [Peng-Zhu-Gao-Wang-Yuan'25]

$$\mathcal{H} = \{\mathbf{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d} \mid \|\mathbf{X}\|_{\mathrm{F}}^2 - 1 = 0\}$$

Sparse constraint + an additional constraint

$$\begin{aligned} \min_{X \in \mathbb{R}^m} \quad & f(X) \\ \mathrm{s.\,t.} \quad & h(X) = 0 \\ & X \in \mathcal{C}_s = \{X \in \mathbb{R}^m \mid \|X\|_0 = s\} \end{aligned}$$

Sparse + Frobenius sphere

Gene expression patterns and financial applications [Beck-Vaisbourd'16 JOTA]

$$\mathcal{H} = \{X \in \mathbb{R}^m \mid \|X\|_{\mathrm{F}}^2 - 1 = 0\}$$

Sparse + Stiefel manifold

Sparse principal component analysis problems [Chen-Huang'26]

$$\mathcal{H} = \{X \in \mathbb{R}^{n \times p} \mid X^\top X - I_p = 0\}$$

Optimization over intersection of manifolds

Additional structures imposed on the general manifold $\mathcal{M}$

$$\begin{aligned} \min_{X \in \mathcal{E}} \quad & f(X) \\ \mathrm{s.\,t.} \quad & X \in \mathcal{H} \cap \mathcal{M} \end{aligned}$$

where $\mathcal{H} = \{X \in \mathcal{E} \mid h(X) = 0\}$

Challenges introduced by the coupled constraints

- Lack of manifold structure on $\mathcal{H} \cap \mathcal{M}$
- Geometry of the coupled region $\mathcal{H} \cap \mathcal{M}$
  - Characterization of the tangent cone $\mathrm{T}_{\mathcal{H} \cap \mathcal{M}}(X)$
- Difficulty of preserving the feasibility $X \in \mathcal{H} \cap \mathcal{M}$
  - Absence of closed-form $\mathcal{P}_{\mathcal{H} \cap \mathcal{M}}(\cdot)$
  - Iterates of alternating projections [von Neumann'50; Lewis-Malick'08]

Existing algorithms and motivations

Optimization on $\mathcal{M}$ with equality constraints

$$\begin{aligned} \min_{X\in\mathcal{M}} \quad & f(X) \\ \mathrm{s.\,t.} \quad & h(X) = 0 \end{aligned}$$

- Constraint qualifications and optimality conditions [Bergmannand-Herzog'19 SIOPT; Andreani-Couto-Ferreira-Haeser'24 SIOPT; Andreani-Couto-Ferreira-Haeser-Prudente'26 SIOPT]
- Riemannian augmented Lagrangian methods [Liu-Boumal'20; Zhou-Bao-Ding-Zhu'23 MP]
- Riemannian SQP methods [Schiela-Ortiz'21 SIOPT; Obara-Okuno-Takeda'22 SIOPT]
- Riemannian interior point methods [Lai-Yoshise'24 JOTA]
- Retractions by alternating projections [Chen-He-Huang'26]
- Exact penalty methods when $\mathcal{M}$ is convex [Xiao-Liu-Toh'24 MOR; Xiao-Tang-Wang-Toh'25]

Discussions

☹ A sequence of subproblems should be solved, e.g.,

$$\min_{X\in\mathcal{M}} f(X) + \langle \lambda, h(X)\rangle + \frac{\beta}{2}\|h(X)\|^2$$

Two subtasks

Optimization over the intersection

$$\begin{aligned} \min_{X \in \mathcal{E}} \quad & f(X) \\ \text{s.t.} \quad & X \in \mathcal{H} \cap \mathcal{M} \end{aligned}$$

I. Feasibility

Find points $X \in \mathcal{H} \cap \mathcal{M}$

II. Optimality

Decrease f over $\mathcal{H} \cap \mathcal{M}$

Alternating projections

Method of alternating projections [von Neumann'50]

MAP Iteration

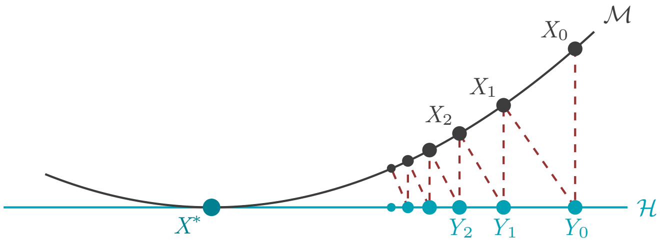
$$X_{k+1} \in \mathcal{P}_{\mathcal{M}}\big(\mathcal{P}_{\mathcal{H}}(X_k)\big)$$

$\Longrightarrow$

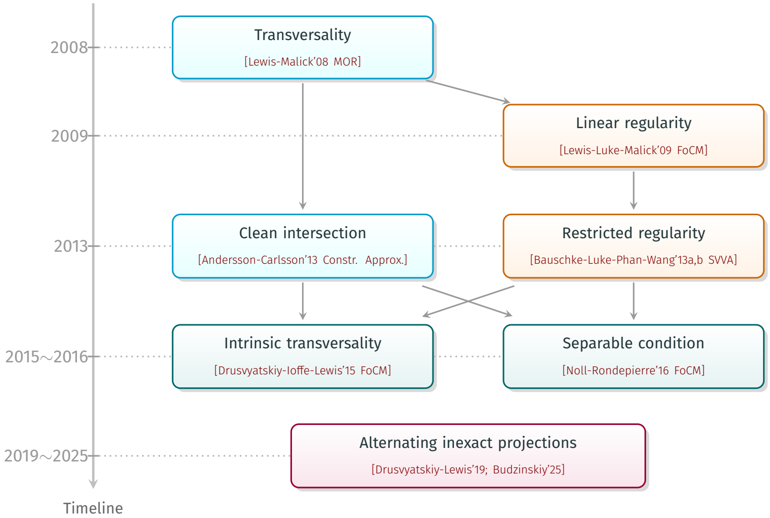
Feasibility Problem

Find a point $X^* \in \mathcal{H} \cap \mathcal{M}$

Illustration



Developments and intersection conditions



Motivation of the notions

Transversality

$\mathrm{T}_{\mathcal{H}}(X) + \mathrm{T}_{\mathcal{M}}(X) \;=\; \mathcal{E}$

→

Clean intersection

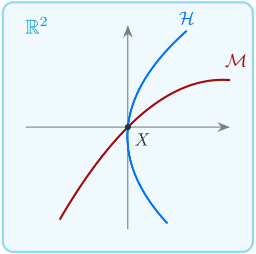
$\mathrm{T}_{\mathcal{H}\cap\mathcal{M}}(X) = \mathrm{T}_{\mathcal{H}}(X) \cap \mathrm{T}_{\mathcal{M}}(X)$

→

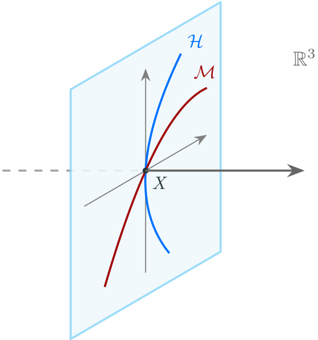
Intrinsic transversality

$\mathrm{N}_{\mathcal{H}}(X)$ and $\mathrm{N}_{\mathcal{M}}(X)$ are "separable"

Open question [Ioffe'17 JOTA]



Trivial lifting



$\mathrm{T}_{\mathcal{H}}(X) + \mathrm{T}_{\mathcal{M}}(X) = \mathbb{R}^2$

Transversality holds

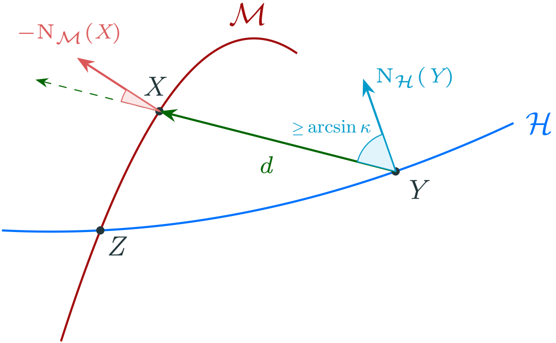
$\mathrm{T}_{\mathcal{H}}(X) + \mathrm{T}_{\mathcal{M}}(X) \subsetneq \mathbb{R}^3$

Transversality fails

The sets $\mathcal{H}$ and $\mathcal{M}$ are ***intrinsically transversal*** at $z \in \mathcal{H} \cap \mathcal{M}$ if there exist $\kappa \in (0, 1]$ and a neighborhood $\mathcal{B}$ of Z such that for all $X \in (\mathcal{M} \cap \mathcal{B}) \setminus \mathcal{H}$ and $Y \in (\mathcal{H} \cap \mathcal{B}) \setminus \mathcal{M}$ with $d = \frac{X - Y}{\|X - Y\|}$, one has

$$\max\left\{ \operatorname{dist}\left(d, -\mathrm{N}_{\mathcal{M}}(X)\right),\ \operatorname{dist}\left(d, \mathrm{N}_{\mathcal{H}}(Y)\right) \right\} \geq \kappa$$

Illustration



A geometric framework

$$\min_{X \in \mathcal{E}} f(X) \quad \text{s.t. } X \in \mathcal{H} \cap \mathcal{M}$$ (Intrinsically Transversal)

$$\min_{X \in \mathcal{M}} f(X) \quad \text{s.t. } h(X) = 0$$

I. Feasibility

Find points $X \in \mathcal{H} \cap \mathcal{M}$

Perspective I on IT

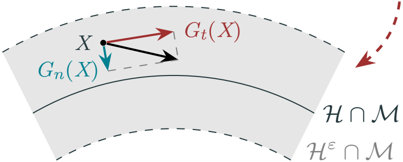
$$G_n(X) \in \mathrm{T}_{\mathcal{M}}(X) \Rightarrow \mathrm{dist}(X, \mathcal{H}) \downarrow$$

II. Optimality

Decrease f over $\mathcal{H} \cap \mathcal{M}$

Perspective II on IT

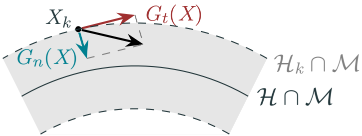
$$G_t(X) \in \mathrm{T}_{\mathcal{M}}(X) \Rightarrow f(X) \downarrow$$



Perturbed manifolds

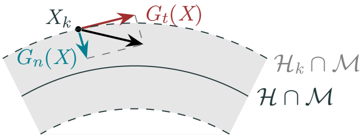
$$\mathcal{H}(Y) := \{X \in \mathcal{E} \mid h(X) = h(Y)\}$$

$$\mathcal{H}_k := \mathcal{H}(X_k)$$



Update rule

$$\boxed{X_{k+1} = \mathrm{R}^{\mathcal{M}}_{X_k}\left(\alpha_k G_n(X_k) + \beta_k G_t(X_k)\right)}$$



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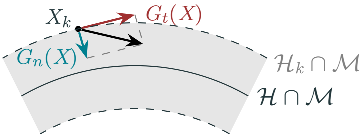
Important observation

Given two manifolds $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{E}$ with a point $X \in \mathcal{X}_1 \cap \mathcal{X}_2$. Then for any $d \in \mathrm{N}_{\mathcal{X}_2}(X)$ and $\eta \in \mathcal{E}$, it holds that

$$\left\langle \mathcal{P}_{\mathrm{T}_{\mathcal{X}_1}(X)}(d), \mathcal{P}_{\mathrm{T}_{\mathcal{X}_1 \cap \mathcal{X}_2}(X)}(\eta) \right\rangle = 0$$

A geometric framework (cont'd)

Perturbed manifolds



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$$\left\langle \mathcal{P}_{\mathrm{T}_{\mathcal{X}_1}(X)}(d), \mathcal{P}_{\mathrm{T}_{\mathcal{X}_1 \cap \mathcal{X}_2}(X)}(\eta) \right\rangle = 0$$

$$\boxed{G_n(X_k) = \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X_k)}(d_k)}$$

$$\boxed{G_t(X_k) = \mathcal{P}_{\mathrm{T}_{\mathcal{H}_k \cap \mathcal{M}}(X_k)}(\eta_k)}$$

Perturbed manifolds

$$\mathcal{H}(Y) := \{X \in \mathcal{E} \mid h(X) = h(Y)\}$$

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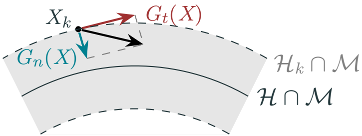

Update rule

$$X_{k+1} = \mathrm{R}^{\mathcal{M}}_{X_k}\left(\alpha_k \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X_k)}(d_k) + \beta_k \mathcal{P}_{\mathrm{T}_{\mathcal{H}_k \cap \mathcal{M}}(X_k)}(\eta_k)\right)$$

Important observation

Given two manifolds $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{E}$ with a point $X \in \mathcal{X}_1 \cap \mathcal{X}_2$. Then for any $d \in \mathrm{N}_{\mathcal{X}_2}(X)$ and $\eta \in \mathcal{E}$, it holds that

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$$G_n(X_k) = \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X_k)}(d_k)$$

$$G_t(X_k) = \mathcal{P}_{\mathrm{T}_{\mathcal{H}_k \cap \mathcal{M}}(X_k)}(\eta_k)$$

Orthogonal tangent directions

Bouliganded tangent cone tangent space for a manifold

$$\mathrm{T}_{\mathcal{X}}(X) = \{\eta \in \mathcal{E} \mid \exists t_i \to 0, \text{ s. t. } \operatorname{dist}(X + t_i\eta, \mathcal{X}) = o(t_i)\}$$

Given $X \in \mathcal{X}_1 \cap \mathcal{X}_2$, it holds that

$$\mathrm{T}_{\mathcal{X}_1 \cap \mathcal{X}_2}(X) \subseteq \mathrm{T}_{\mathcal{X}_1}(X) \cap \mathrm{T}_{\mathcal{X}_2}(X) \quad \text{and} \quad \mathrm{N}_{\mathcal{X}_1 \cap \mathcal{X}_2}(X) \supseteq \mathrm{N}_{\mathcal{X}_1}(X) + \mathrm{N}_{\mathcal{X}_2}(X)$$

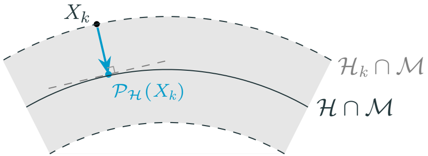
$\mathcal{M}_r$ as a smooth manifold based on the SVD $X = U\Sigma V^\top$

$$\mathrm{T}_{\mathcal{M}_r}(X) = \left\{[U\ U_\perp] \begin{bmatrix} \mathbb{R}^{r\times r} & \mathbb{R}^{r\times(n-r)} \\ \mathbb{R}^{(m-r)\times r} & 0^{(m-r)\times(n-r)} \end{bmatrix} [V\ V_\perp]^\top \right\}$$

$$\mathrm{N}_{\mathcal{M}_r}(X) = \left\{[U\ U_\perp] \begin{bmatrix} 0^{r\times r} & 0^{r\times(n-r)} \\ 0^{(m-r)\times r} & \mathbb{R}^{(m-r)\times(n-r)} \end{bmatrix} [V\ V_\perp]^\top \right\}$$

Tangent directon toward feasibility

$$G_n(X_k) = \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X_k)}(d_k)$$
$$d_k \in \mathrm{N}_{\mathcal{H}_k}(X_k)$$



Construction of $d_k \in \mathrm{N}_{\mathcal{H}_k}(X_k) = \mathrm{span}\{\nabla h(X_k)\}$

- Projection direction $\tilde{d}_k = X_k - \mathcal{P}_{\mathcal{H}}(X_k)$
- Gauss–Newton direction $d_k = -\nabla h(X_k)(\nabla h(X_k)^\top \nabla h(X_k))^{-1} h(X_k)$

$$d_k = \min_{d \in \mathrm{N}_{\mathcal{H}_k}(X_k)} \|h(X_k) + \nabla h(X_k)^\top d\|^2$$

- First-order surrogate $\|\tilde{d}_k - d_k\| = \mathcal{O}(\|\tilde{d}_k\|^2)$

Tangent directon toward feasibility

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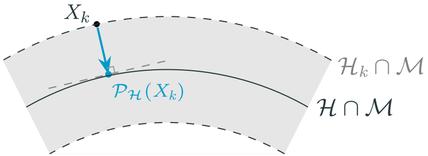
$$d_k \in \mathrm{N}_{\mathcal{H}_k}(X_k)$$



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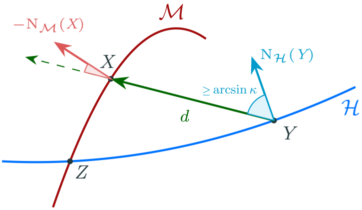
- First-order surrogate $\|\tilde{d}_k - d_k\| = \mathcal{O}(\|\tilde{d}_k\|^2)$

Can $\mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X_k)}(d_k)$ still guarantee $\mathrm{dist}(X_k, \mathcal{H}) \downarrow$?

Perspective I on intrinsic transversality

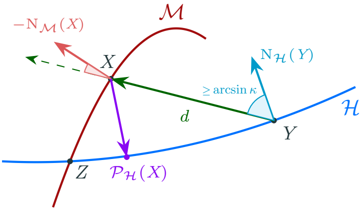
Intrinsic transversality is defined via all (X, Y) pairs around Z

$$\max\{\mathrm{dist}(d, -\mathrm{N}_{\mathcal{M}}(X)),\ \mathrm{dist}(d, \mathrm{N}_{\mathcal{H}}(Y))\} \geq \kappa$$



Intrinsic transversality is defined via **all** (X, Y) pairs around Z

$$\max\{\mathrm{dist}(d, -\mathrm{N}_{\mathcal{M}}(X)),\ \mathrm{dist}(d, \mathrm{N}_{\mathcal{H}}(Y))\} \geq \kappa$$



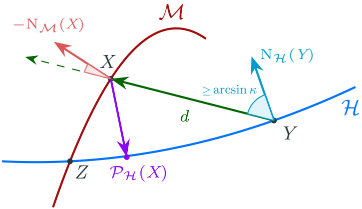
Projection-based transversality $\mathcal{H}, \mathcal{M}$ are C^2 submanifolds

There exists $\kappa' \in (0, 1]$ such that for the **specific** pairs, it holds that

$$(X, \mathcal{P}_{\mathcal{H}}(X)) \Rightarrow \mathrm{dist}\left(d, -\mathrm{N}_{\mathcal{M}}(X)\right) \geq \kappa' \quad (Y, \mathcal{P}_{\mathcal{M}}(Y)) \Rightarrow \mathrm{dist}\left(d, \mathrm{N}_{\mathcal{H}}(Y)\right) \geq \kappa'$$

Intrinsic transversality is defined via
all (X, Y) pairs around Z

$$\max\{\mathrm{dist}(d, -\mathrm{N}_{\mathcal{M}}(X)),\ \mathrm{dist}(d, \mathrm{N}_{\mathcal{H}}(Y))\} \geq \kappa$$



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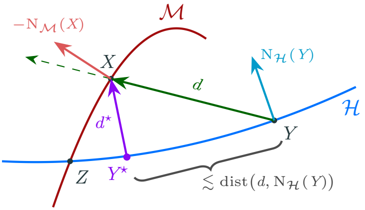
Theorem

Intrinsic transversality $\Longleftrightarrow$ Projection-based transversality

Theorem $\mathcal{H}, \mathcal{M}$ are C^2 submanifolds

Intrinsic transversality $\iff$ Projection-based transversality

Nontrivial direction "$\Longleftarrow$"



- Assume IT fails: (X, Y) with d nearly normal to both $\mathcal{H}, \mathcal{M}$; $Y^\star := \mathcal{P}_{\mathcal{H}}(X)$
- Prox-regularity of manifolds gives $\|Y - Y^\star\| \lesssim \mathrm{dist}(d, \mathrm{N}_{\mathcal{H}}(Y))$, and thus

$$\begin{aligned}\mathrm{dist}\big(d^\star, \mathrm{N}_{\mathcal{M}}(X)\big) &\le \mathrm{dist}\big(d, \mathrm{N}_{\mathcal{M}}(X)\big) + C_1\|d - d^\star\| \\ &\le \mathrm{dist}\big(d, \mathrm{N}_{\mathcal{M}}(X)\big) + C_2\ \mathrm{dist}\big(d, \mathrm{N}_{\mathcal{H}}(Y)\big)\end{aligned}$$

- IT fails $\iff$ both terms are small $\implies$ $(X, Y^\star)$ leads to a contradiction

Tangent direction I: projected Gauss–Newton direction

$$\begin{array}{ll} \min\limits_{X \in \mathcal{E}} & f(X) \\ \text{s.t.} & X \in \mathcal{H} \cap \mathcal{M} \end{array}$$

$$G_n(X_k) = \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X_k)}(d_k)$$
$$d_k = -\nabla h(X_k)(\nabla h(X_k)^\top \nabla h(X_k))^{-1} h(X_k)$$

Projection-based transversality

$$\|\mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X_k)}(d_k)\| \geq \kappa' \|d_k\|$$

Non-degeneracy of ∇h

$$\|d_k\| \geq \sigma_{\max}^{-1}(\nabla h(X_k)) \|h(X_k)\|$$

Optimality analysis

$$\begin{aligned} G_n(X_k) = 0 \iff \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X_k)}(d_k) = 0 \\ \implies h(X_k) = 0 \end{aligned}$$

Perspective II on intrinsic transversality

$$
\begin{tikzcd}
\begin{array}{c}\text{Transversality}\\ \mathrm{T}_{\mathcal{H}}(X) + \mathrm{T}_{\mathcal{M}}(X) \;=\; \mathcal{E}\end{array} \arrow[r] & \begin{array}{c}\text{Clean intersection}\\ \mathrm{T}_{\mathcal{H}\cap\mathcal{M}}(X) = \mathrm{T}_{\mathcal{H}}(X) \cap \mathrm{T}_{\mathcal{M}}(X)\end{array} \arrow[r] & \begin{array}{c}\text{Intrinsic transversality}\\ \mathrm{N}_{\mathcal{H}}(X) \text{ and } \mathrm{N}_{\mathcal{M}}(X) \text{ are ``separable''}\end{array} \arrow[l, bend left, dashed, "\text{Open question [Ioffe'17 JOTA]}"]
\end{tikzcd}
$$

Transversality	Clean intersection	Intrinsic transversality
$\mathrm{T}_{\mathcal{H}}(X) + \mathrm{T}_{\mathcal{M}}(X) = \mathcal{E}$	$\mathrm{T}_{\mathcal{H}\cap\mathcal{M}}(X) = \mathrm{T}_{\mathcal{H}}(X) \cap \mathrm{T}_{\mathcal{M}}(X)$	$\mathrm{N}_{\mathcal{H}}(X)$ and $\mathrm{N}_{\mathcal{M}}(X)$ are "separable"

Open question [Ioffe'17 JOTA]

Theorem answers the open question

Let $\mathcal{H}, \mathcal{M} \subseteq \mathcal{E}$ be C^1 manifolds, and they are intrinsically transversal at X. Then they intersect *cleanly* near X: there exists a neighborhood $\mathcal{B}$ of X such that $\mathcal{H} \cap \mathcal{M} \cap \mathcal{B}$ is a C^1 manifold and, for all $Y \in \mathcal{B}$, it holds that

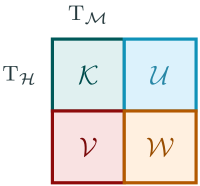
$$\boxed{\mathrm{T}_{\mathcal{H}\cap\mathcal{M}}(Y) = \mathrm{T}_{\mathcal{H}}(Y) \cap \mathrm{T}_{\mathcal{M}}(Y)}$$

Proof sketch

Theorem $\mathcal{H}, \mathcal{M}$ are C^1 submanifolds, intrinsically transversal at Z

$\mathcal{H} \cap \mathcal{M}$ is a C^1 submanifold near Z with $\mathrm{T}_{\mathcal{H}\cap\mathcal{M}} = \mathrm{T}_{\mathcal{H}} \cap \mathrm{T}_{\mathcal{M}}$

- Intersection regularities are invariant under diffeomorphisms
- Flatten $\mathcal{H}$ into a linear space; orthogonally decompose $\mathcal{E}$ along $(\mathrm{T}_{\mathcal{H}}, \mathrm{T}_{\mathcal{M}})$



$\mathcal{E} = \mathcal{K} \oplus \mathcal{U} \oplus \mathcal{V} \oplus \mathcal{W}$

$\mathcal{K} = \mathrm{T}_{\mathcal{H}} \cap \mathrm{T}_{\mathcal{M}}$ and $\mathcal{W} = (\mathrm{T}_{\mathcal{H}} + \mathrm{T}_{\mathcal{M}})^{\perp}$

Graph representation $\begin{cases} \mathcal{M} = \{(k, a(k,v), v, b(k,v)) \mid (k,v) \in \mathcal{K} \oplus \mathcal{V}\} \\ \mathcal{H} = \{(k, u, 0, 0) \mid (k,u) \in \mathcal{K} \oplus \mathcal{U}\} \end{cases}$

★ Intrinsic transversality implies that $b(k,0) \equiv 0$ for all $k \in \mathcal{K}$ small enough

- $\mathcal{H} \cap \mathcal{M} = \{(k, a(k,0), 0, 0) \mid k \in \mathcal{K}\}$ is a graph on $\mathcal{K}$, thus C^1 **embedded**

Tangent direction II: projected anti-gradient direction

$$\begin{array}{ll} \min\limits_{X \in \mathcal{E}} & f(X) \\ \mathrm{s.\,t.} & X \in \mathcal{H} \cap \mathcal{M} \end{array}$$

$$G_t(X_k) = \mathcal{P}_{\mathrm{T}_{\mathcal{H}_k \cap \mathcal{M}}(X_k)}\left(-\nabla f(X_k)\right)$$

Optimality analysis

$$\begin{aligned} G_t(X_k) = 0 &\iff \mathcal{P}_{\mathrm{T}_{\mathcal{H}_k \cap \mathcal{M}}(X_k)}\left(-\nabla f(X_k)\right) = 0 \\ &\iff -\nabla f(X_k) \in \mathrm{N}_{\mathcal{H}_k \cap \mathcal{M}}(X_k) \end{aligned}$$

Descent property

$$\left\langle \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X)}\left(\nabla f(X)\right), \mathcal{P}_{\mathrm{T}_{\mathcal{H} \cap \mathcal{M}}(X)}\left(-\nabla f(X)\right) \right\rangle = -\left\| \mathcal{P}_{\mathrm{T}_{\mathcal{H} \cap \mathcal{M}}(X)}\left(-\nabla f(X)\right) \right\|^2$$

Projection onto the tangent space

Proposition $S(X) = \mathrm{T}_{\mathcal{H}(X)}(X) \cap \mathrm{T}_{\mathcal{M}}(X)$

Assume that $\mathcal{H}(X)$ and $\mathcal{M}$ are intrinsically transversal at X, and let $\Phi_{\mathcal{M}}(X) := \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X)} \circ \mathrm{D}h_X^*$. Then for any $\xi \in \mathcal{E}$,

$$\mathcal{P}_{S(X)}(\xi) = \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X)}(\xi) - \Phi_{\mathcal{M}}(X)\big(\mathrm{D}h_X \circ \Phi_{\mathcal{M}}(X)\big)^{\dagger}\mathrm{D}h_X\big(\mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X)}(\xi)\big)$$

When they are transversal, the pseudoinverse reduces to the ordinary inverse

☺ Computing $G_t(X) = \mathcal{P}_{\mathrm{T}_{\mathcal{H}\cap\mathcal{M}}(X)}(-\nabla f(X))$ is tractable

Projection onto the tangent space (cont'd)

Manifold	Level set of h	$\mathcal{P}_{\mathrm{T}_{\mathcal{H}\cap\mathcal{M}}(X)}$	Format
$\mathcal{M}_r$	$\{X \in \mathbb{R}^{m\times n} \mid \mathcal{A}(X) = b\}$	[Yang-Gao-Yuan'25]	
$\mathcal{M}_r$	$\mathcal{H}$ is orthogonally invariant	[Yang-Gao-Yuan'26 MP]	
$\mathcal{M}_r$	$\mathcal{H}$ is hyperbolic	Appendix A	low-rank
$\mathcal{S}_r(n)$	$\{X \in \mathcal{S}(n) \mid \Vert X\Vert_{\mathrm{F}}^2 = 1\}$	[Li-Xiu-Zhou'20 JOTA]	
$\mathcal{M}_{\mathbf{r}}^{\mathrm{ht}}$	$\{\mathbf{X} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d} \mid \Vert\mathbf{X}\Vert_{\mathrm{F}}^2 = 1\}$	[Yang-Gao-Yuan'25]	
$\mathcal{M}_{\mathbf{r}}^{\mathrm{tt}}$	Reshaped Stiefel manifold	Appendix B	
$\mathcal{C}_s$	$\{X \in \mathbb{R}^{n} \mid \Vert X\Vert_{\mathrm{F}}^2 = 1\}$	[Beck-Hallak'16 MOR]	sparse
$\mathcal{C}_s$	$\{X \in \mathbb{R}^{n\times p} \mid X^\top X = I_p\}$	[Chen-Huang'26]	
$\mathcal{M}$	$\mathcal{H}\cap\mathcal{M}$ is intrinsically transversal	**Our proposition**	general

Algorithm and convergence analysis

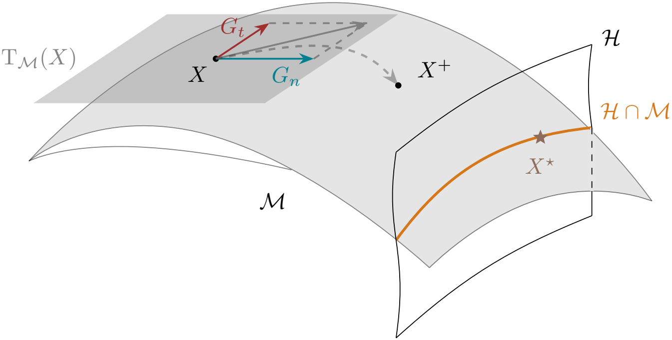
A Geometric method via Orthogonal Tangent Directions (GOTD)

$$d_k = -\nabla h(X_k)(\nabla h(X_k)^\top \nabla h(X_k))^{-1} h(X_k)$$

$$\eta_k = -\nabla f(X_k)$$

$$X_{k+1} = \mathrm{R}^{\mathcal{M}}_{X_k}\left(\alpha_k \mathcal{P}_{\mathrm{T}_{\mathcal{M}}(X_k)}(d_k) + \beta_k \mathcal{P}_{\mathrm{T}_{\mathcal{H}_k \cap \mathcal{M}}(X_k)}(\eta_k)\right)$$

Illustration



Assumptions

- There exists a constant $\rho > 0$ such that ∇h has full rank on
$$\mathcal{K}(\rho) := \{X \in \mathbb{R}^m \mid \|h(X)\| \leq \rho\}$$
- The intersection $\mathcal{H}(Y) \cap \mathcal{M}$ is intrinsically transversal for $Y \in \mathcal{K}(\rho)$
- Riemannian L-smoothness of f on $\mathcal{M}$

Optimality

- The condition $\|G_t(X)\| \leq \varepsilon$ reveals $\|\mathcal{P}_{\mathrm{T}_{\mathcal{H}(X)\cap\mathcal{M}}}(-\nabla f(X))\| \leq \varepsilon$
- When $\varepsilon = 0$ and $h(X) = 0$, the first-order optimality holds
$$\mathcal{P}_{\mathrm{T}_{\mathcal{H}\cap\mathcal{M}(X)}}(-\nabla f(X)) = 0$$

Theorem

In GOTD, with appropriate constant step sizes α, β, the iterates $\{X_k\}$ satisfy
$$\frac{1}{K}\sum_{k=1}^{K}\Big(\|G_t(X_k)\|^2 + \mathrm{dist}(X_k, \mathcal{H})\Big) = \mathcal{O}(\frac{1}{K})$$

Numerical experiments

Low-rank approximation of spherical data

Approximate $A \in \mathcal{H} = \mathrm{Ob}(m, n)$ with a low-rank matrix [Chu et al., 2005 SIAMX]

$$
\begin{aligned}
\min_{X \in \mathbb{R}^{m \times n}} \quad & \frac{1}{2} \left\| \mathcal{P}_\Omega (X - A) \right\|^2 \\
\text{s. t.} \quad & \mathrm{diag}(XX^\top) - \mathbf{1} = 0 \\
& \mathrm{rank}(X) = r
\end{aligned}
$$

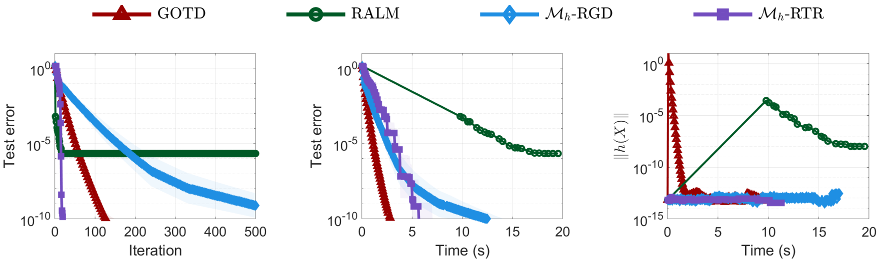
Methods based on Manopt v8.0

- GOTD (ours)
- Riemannian augmented Lagrangian method (RALM) [Liu-Boumal'20]
- Space-decoupling parameterization $\mathcal{M}_h$ [Yang-Gao-Yuan'26 MP]

Settings

- Synthetic $A \in \mathcal{M}_{\leq r} \cap \mathrm{Ob}(m, n)$, $(m, n) = (5000, 6000)$
- rank parameter $r = r^* = 10$, oversampling factor $\mathrm{OS} = 6$

Test on synthetic data



- ☺ RALM and $\mathcal{M}_h$-RTR exhibit better iteration complexity, but they should solve a subproblem for each iterate
- ☺ GOTD recovers the matrix faster than RALM and methods on $\mathcal{M}_h$
- ☺ Geometric methods achieve better precision

Low-rank approximation of hyperbolic embeddings

Approximate $A \in \mathbb{H}_n^m$ under hyperbolic distance [Nickel-Kiela'17]

$$\begin{aligned} \min_{X \in \mathbb{R}^{(n+1)\times m}} \quad & \sum_{i=1}^{m} \operatorname{arccosh}(-\langle X_i, A_i \rangle_{\mathcal{L}})^2 \\ \text{s.t.} \quad & X \in \mathbb{H}_n^m \\ & \operatorname{rank}(X) = r+1 \end{aligned}$$

$$\mathcal{H} = \mathbb{H}_n^m := \{X \in \mathbb{R}^{(n+1)\times m} \mid \operatorname{diag}(X^\top \operatorname{Diag}(-1, 1, \ldots, 1)X) - \mathbf{1} = \mathbf{0}\}$$

Methods

- GOTD (ours)
- Riemannian augmented Lagrangian method (RALM) [Liu-Boumal'20]
- Riemannian parameterization (RGD, RTR) [Jawanpuria-Meghwanshi-Mishra'19 CDC]

Settings

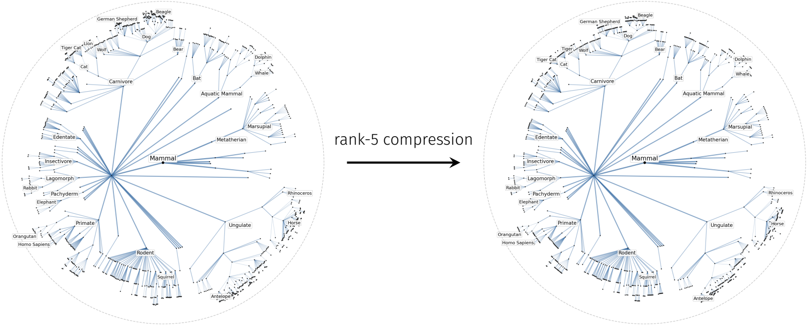
- The mammal subtree of the WordNet database
- $(n, m) = (300, 1170)$, rank parameter $r = 5, 10, 20$

Results of low-rank hyperbolic embeddings

Evaluation on mean average precision (original embeddings: 0.9385)

Rank	GOTD (Ours)	RALM	RGD	RTR
$r = 5$	0.8869	0.6637	0.8881	**0.8883**
$r = 10$	**0.9039**	0.8117	0.8961	0.9024
$r = 20$	**0.9003**	0.8463	0.8989	0.8981

Illustration on 2-dimensional Poincaré ball



Spatially localized solutions of the Schrödinger's equation

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times p}} \quad & \operatorname{tr}(X^\top H X) \\ \text{s.t.} \quad & X \in \operatorname{St}(n, p) \\ & X \in \mathcal{M} = \{ Y \in \mathbb{R}^{n \times p} \mid \|Y\|_0 = s \} \end{aligned} \xrightarrow{\text{penalty}} \begin{aligned} \min_{X \in \mathbb{R}^{n \times p}} \quad & \operatorname{tr}(X^\top H X) + \mu \|X\|_1 \\ \text{s.t.} \quad & X \in \operatorname{St}(n, p) \end{aligned}$$

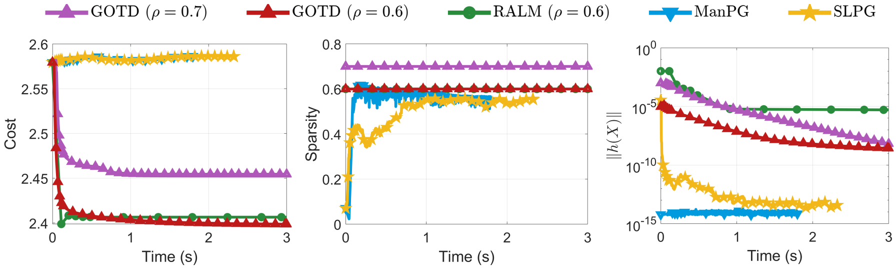
Methods

- GOTD (ours)
- Riemannian augmented Lagrangian method (RALM) [Liu-Boumal'20]
- Manifold proximal gradient method (ManPG) [Chen-Ma-So-Zhang'20]
- Sequential linearized proximal gradient (SLPG) [Liu-Xiao-Yuan'24]

Settings

- H is symmetric, discretizing the Hamiltonian $-\frac{1}{2}\partial_x^2$
- $(n, p) = (256, 15)$, sparsity $\rho := (np - \|X\|_0)/np \in \{0.6,\, 0.7\}$

Comparison on cost, sparsity, and feasibility



- ☺ GOTD finds lower cost with lower density than ManPG and SLPG
- ☺ GOTD achieves better feasibility than RALM

Conclusion

$$\min_{X\in\mathcal{E}} f(X) \quad \text{s.t. } X \in \mathcal{H}\cap\mathcal{M}$$ (Intrinsically Transversal)

$$\min_{X\in\mathcal{M}} f(X) \quad \text{s.t. } h(X) = 0$$

I. Feasibility

Find points $X \in \mathcal{H} \cap \mathcal{M}$

Perspective I on IT

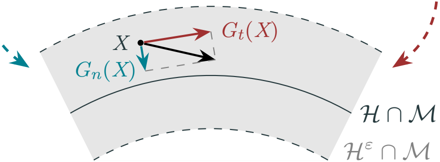
$$G_n(X) \in \mathrm{T}_{\mathcal{M}}(X) \Rightarrow \mathrm{dist}(X, \mathcal{H}) \downarrow$$

II. Optimality

Decrease f over $\mathcal{H} \cap \mathcal{M}$

Perspective II on IT

$$G_t(X) \in \mathrm{T}_{\mathcal{M}}(X) \Rightarrow f(X) \downarrow$$



Conclusion and future work

- ☺ Efficient algorithm exploiting the geometry [Yang-Gao-Yuan'25]

$$\mathrm{T}_{\mathcal{H}\cap\mathcal{M}}(X) = \mathrm{T}_{\mathcal{H}}(X) \cap \mathrm{T}_{\mathcal{M}}(X)$$

- ? Second-order development beneficial from the property [Yang-Gao-Yuan'25]

$$\mathrm{T}^2_{\mathcal{H}\cap\mathcal{M}}(X;\eta) = \mathrm{T}^2_{\mathcal{H}}(X;\eta) \cap \mathrm{T}^2_{\mathcal{M}}(X;\eta)$$

- ? Reasonable extension to multiple manifolds $\mathcal{M}$, $\mathcal{H}_1, \ldots, \mathcal{H}_N$
- ? Technical adaptation to non-smooth sets, e.g., $\mathcal{M} = \mathcal{M}_{\leq r}$, $\mathcal{C}_{\leq s}$
- ? Stochastic oracles and variance-reduction algorithms

References

Yan Yang, Bin Gao, Ya-xiang Yuan. *A space-decoupling framework for optimization on bounded-rank matrices with orthogonally invariant constraints.* Mathematical Programming (2026)

Yan Yang, Bin Gao, Ya-xiang Yuan. Variational analysis of determinantal varieties. arXiv:2511.22613 (2025)

Thanks for your attention!

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