Variational analysis of determinantal varieties

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Outline

1. Background: low-rank optimization
2. Variational analysis of low-rank sets
3. First- and second-order tangent sets
4. Tangent sets bridge optimization landscapes
5. Second-order optimality on bounded-rank matrices

Background: low-rank optimization

Low-rank problems: a geometric view

Optimization on the fixed-rank manifold $r < \min\{m, n\}$

$$\min \ f(X) \quad \text{s.t. } X \in \mathcal{M}_r := \{X \in \mathbb{R}^{m\times n} \mid \operatorname{rank}(X) = r\}$$

- $\mathcal{M}_r$ is a smooth manifold [Helmke-Shayman'95]
- Riemannian optimization algorithms [Vandereycken'13; He-Li-Jiang-Ma-Zhang'26]
- $\mathcal{M}_r$ is not closed

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Optimization on the set of bounded-rank matrices $r < \min\{m, n\}$

$$\min \; f(X) \quad \text{s.t. } X \in \mathcal{M}_{\le r} := \{X \in \mathbb{R}^{m\times n} \mid \operatorname{rank}(X) \le r\}$$

- $\mathcal{M}_{\le r}$ is closed
- $\mathcal{M}_{\le r}$ is a real-algebraic variety: determinantal variety [Harris'92]
- Finding first-order stationary points [Schneider-Uschmajew'15; Levin-Kileel-Boumal'22; '24; Olikier-Absil'22; '23; '24]

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A unified variational analysis for low-rank sets ?

Applications of low-rank optimization

$$\boxed{\begin{array}{ll} \min & f(X) \\ \text{s.t.} & X \in \mathcal{M}_{\leq r} \end{array}}$$

- Recommendation system: movie ratings [Frolov-Oseledets'17]
- Hyperspectral Images [Zhang-He-Zhang-Shen-Yuan'13; Zhuang-Fu-Ng'21]
- Image and video inpainting [Bertalmio-Sapiro-Caselles-Ballester'00; Fu-Ruan-Luo-An-Jin'21; Luo-Zhao-Li-Ng-Meng'23]
- EEG (brain signals) data [Mørup-Hansen-Herrmann-Parnas-Arnfred'06; Kong-Kong-Fan-Zhao-Cichoki'17]
- Magnetic resonance imaging (MRI) [Banco-Aeron-Hoge'16; Choi-Bao-Zhang'18; Fessler'20]
- Data analysis, e.g., weather forecast [Loucheur-Absil-Journée'23] and exoplanet detection [Daglayan-Vary-Absil-Cantalloube-Christiaens-Gillis-Jacques-Leplat-Absil'24]

Low rank + additional constraints

Bounded-rank matrices + an additional constraint

$$\begin{array}{ll} \min & f(X) \\ \text{s.t.} & X \in \mathcal{M}_{\leq r} \cap \mathcal{H} \end{array}$$

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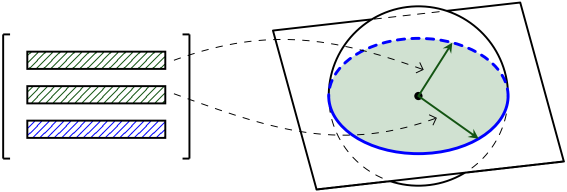
Low rank + oblique manifold

- Low-rank data fitting on sphere [Chu-Del Buono-Lopez-Politi'05 SIMAX]

$$\mathcal{H} = \mathrm{Ob}(m, n) := \{X \in \mathbb{R}^{m\times n} \mid \operatorname{diag}\left(XX^\top\right) - \mathbf{1} = \mathbf{0}\}$$

$\mathbb{R}^{3\times 4}_{\leq 2} \cap \mathrm{Ob}(3, 4) \ni$



Low rank + additional constraints (cont'd)

Low rank + Frobenius sphere

- Approximation of graph similarity matrices [Cason-Absil-Van Dooren'13 LAA]

$$\mathcal{H} = \mathrm{S}_{\mathrm{F}}(m,n) := \{X \in \mathbb{R}^{m \times n} \mid \|X\|_{\mathrm{F}}^2 - 1 = 0\}$$

Low rank + hyperbolic manifold

- Approximation of hyperbolic embeddings [Jawanpuria-Meghwanshi-Mishra'19 CDC]

$$\mathcal{H} = \mathbb{H}^m_{n-1} := \{X \in \mathbb{R}^{m \times n} \mid \mathrm{diag}(X\,\mathrm{Diag}(-1,1,\dots,1)X^\top) - \mathbf{1} = \mathbf{0}\}$$

Low rank + additional constraints (cont'd)

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Low rank + SDPs

- Low-rank semidefinite programming [Journée-Bach-Absil-Sepulchre'10 SIOPT]

$$\begin{aligned} \min \quad & \langle C, X\rangle \\ \text{s.t.} \quad & X \in \mathcal{M}_{\leq r} \cap \mathcal{S}^{+}(n) \cap \mathcal{U} \end{aligned}$$

where $\mathcal{U} = \{X \in \mathbb{R}^{n\times n} \mid \mathcal{A}(X) = b\}$

Low rank + additional constraints (cont'd)

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☆ First- and second-order optimality conditions

Variational analysis of low-rank sets

Challenges

Constraint-coupled optimization

$$\begin{aligned} \min \quad & f(X) \\ \text{s. t.} \quad & X \in \mathcal{M}_{\leq r} \cap \mathcal{H} \end{aligned}$$

Challenges

- Inherent non-smoothness of $\mathcal{M}_{\leq r}$ [Olikier-Absil'22 SVVA; Levin-Kileel-Boumal'23; '25 MP]
- Geometry of the coupled region $\mathcal{M}_{\leq r} \cap \mathcal{H}$
  - Complicated Bouligand-tangent cone
  - Unknown second-order tangent set

Geometry: first-order analysis of determinantal varieties

Tangent and normal cones to $\mathcal{M}_{\leq r}$

- Mordukhovich normal cone [Luke'13]
- Bouligand tangent cone and Frèchet normal cone [Cason-Absil-Van Dooren'13]
- Clarke tangent cone and Clarke normal cone [Hosseini-Uschmajew'19; Li-Song-Xiu'19]
- Continuity of tangent cones [Olikier-Absil'22]

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First-order geometry of $\mathcal{M}_{\le r} \cap \mathcal{H}$

- Mordukhovich normal cone: $\mathcal{H} = \mathrm{Sym}(n)$ [Tam'17]
- Frèchet normal cone: $\mathcal{H} = \mathrm{Sym}(n) \cap \mathcal{B}_i$ [Li-Xiu-Zhou'20 JOTA]

$$\mathcal{B}_1 = \{X \mid \|X\|_{\mathrm{F}}^2 \le 1\},\ \mathcal{B}_2 = \{X \mid -tI_n \preceq X \preceq tI_n\},\ \mathcal{B}_3 = \{X \mid X \succeq 0,\ \mathrm{tr}(X) = 1\}$$

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★ Bouligand tangent cone and Frèchet normal cone:
$\mathcal{H} = \{X \in \mathbb{R}^{m\times n} \mid h(X) = 0\}$ with orthogonally invariant h [Yang-Gao-Yuan'26 MP]

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A unified analysis remains unclear

Algorithm: finding first-order stationary points

- Projected gradient descent framework [Schneider-Uschmajew'15; Olikier-Absil'22;'24]
- Rank-adaptive methods [Olikier-Absil'23;'24; Gao-Peng-Yuan'25]
- Optimizing via a smooth parameterization

$$\begin{tikzcd} \overline{\mathcal{M}} \arrow[d, "\phi"'] \arrow[dr, "\bar{f} := f \circ \phi"] & \\ \mathcal{M}_{\leq r} \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

$$\min \bar{f}(Y) \quad \text{s. t. } Y \in \overline{\mathcal{M}}$$

  - LR factorization [Park-Kyrillidis-Caramanis-Sanghavi'18; Levin-Kileel-Boumal'22]

$$\overline{\mathcal{M}} = \mathbb{R}^{m \times r} \times \mathbb{R}^{n \times r}, \; \phi(L, R) = LR^\top$$

  - Desingularization [Khrulkov-Oseledets'18; Rebjock-Boumal'24]

$$\overline{\mathcal{M}} = \left\{ (X, G) \in \mathbb{R}^{m \times n} \times \mathrm{Gr}(n, n-r) : XG = 0 \right\}, \; \phi(X, G) = X$$

  - Conditions for "$1 \Rightarrow 1$" and "$2 \Rightarrow 1$" [Levin-Kileel-Boumal'25 MP]

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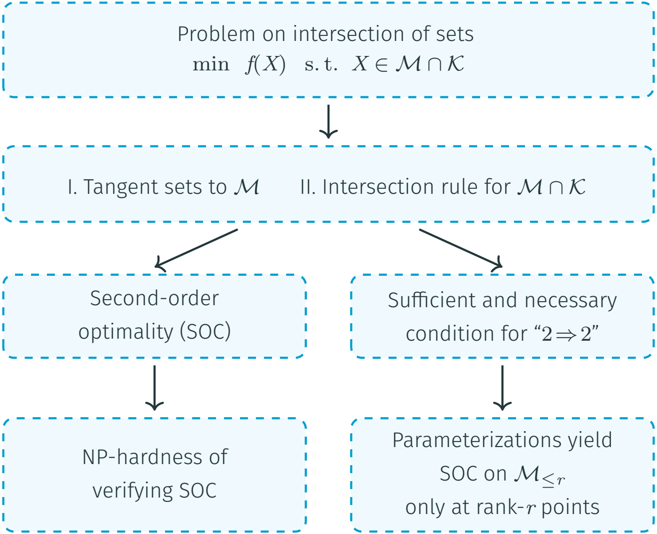
An open question: characterization of "$2 \Rightarrow 2$"

Overview of our contributions

Tangent sets and optimization



Problem on intersection of sets
$\min \quad f(X) \quad \mathrm{s.t.} \quad X \in \mathcal{M} \cap \mathcal{K}$

I. Tangent sets to $\mathcal{M}$

II. Intersection rule for $\mathcal{M} \cap \mathcal{K}$

Second-order optimality (SOC)

Sufficient and necessary condition for “$2 \Rightarrow 2$”

NP-hardness of verifying SOC

Parameterizations yield SOC on $\mathcal{M}_{\leq r}$ only at rank-r points

A unified analysis of tangent sets

Set	Format	First-order	Second-order
$\mathcal{M}$	Assumption 1	Theorem 1	Theorem 1
$\mathcal{M}_{\leq r}$	matrix	[Schneider-Uschmajew'15 SIOPT]	Proposition 1
$\mathcal{M}^{\mathrm{ht}}_{\leq \mathbf{r}}$	hierarchical Tucker	Proposition 2	Proposition 2
$\mathcal{M}^{\mathrm{tc}}_{\leq \mathbf{r}}$	Tucker	[Gao-Peng-Yuan'25 MP]	Proposition 2
$\mathcal{M}^{\mathrm{tt}}_{\leq \mathbf{r}}$	tensor train	[Kutschan'18 LAA]	Proposition 2
$\mathcal{S}_{\leq r}(n)$	symmetric matrix	[Li-Xiu-Zhou'20 JOTA]	Proposition 3
$\mathcal{S}^{+}_{\leq r}(n)$	PSD matrix	[Levin-Kileel-Boumal'25 MP]	Proposition 4
Intersection of sets	Structured set	First-order	Second-order
$\mathcal{M} \cap \mathcal{K}$	Assumption 2	Theorem 2	Theorem 2
$\mathcal{M}_{\leq r} \cap \mathcal{H}$	$\mathcal{H}$ is an affine manifold	[Li-Luo'23 SIOPT]	Appendix C.1
$\mathcal{M}_{\leq r} \cap \mathcal{H}$	$\mathcal{H}$ is orthogonally invariant	[Yang-Gao-Yuan'25]	Appendix C.2
$\mathcal{M}_{\leq r} \cap \mathcal{H}$	$\mathcal{H}$ is hyperbolic	Appendix C.3	Appendix C.3
$\mathcal{S}_{\leq r}(n) \cap \mathcal{U}$	$\mathcal{U} = \{X \mid \|X\|_{\mathrm{F}}^2 = 1\}$	Appendix D.2	Appendix D.2
$\mathcal{S}^{+}_{\leq r}(n) \cap \mathcal{U}$	$\mathcal{U} = \{X \mid \mathcal{A}(X) = b\}$	[Levin-Kileel-Boumal'25 MP]	Appendix D.3

Preliminaries

Bouligand tangent cone

$$\mathrm{T}_{\mathcal{X}}(X) = \{\eta \in \mathcal{E} : \exists t_i \to 0, \text{ s. t. } \operatorname{dist}(X + t_i\eta, \mathcal{X}) = o(t_i)\}$$

Fréchet and Mordukhovich normal cones

$$\begin{tikzcd}\mathrm{T}_{\mathcal{X}}(X) \arrow[r, "\text{polar}"] & \hat{\mathrm{N}}_{\mathcal{X}}(X) \arrow[r, "\lim"] & \mathrm{N}_{\mathcal{X}}(X)\end{tikzcd}$$

Second-order tangent set in the direction η

$$\mathrm{T}^2_{\mathcal{X}}(X, \eta) = \left\{\zeta \in \mathcal{E} : \exists t_i \to 0, \text{ s. t. } \operatorname{dist}(X + t_i\eta + \frac{1}{2}t_i^2\zeta, \mathcal{X}) = o(t_i^2)\right\}$$

Directionally derivative in the direction η

$$h'(X; \eta) := \lim_{t \to 0} \frac{h(X + t\eta) - h(X)}{t}$$

Parabolic second-order directional derivative in the direction (η, ζ)

$$h''(X; \eta, \zeta) := \lim_{t \to 0} \frac{h(X + t\eta + \frac{1}{2}t^2\zeta) - h(X) - th'(X; \eta)}{\frac{1}{2}t^2}$$

First-order geometry of low-rank sets

$\mathcal{M}_s$ as a smooth manifold based on the SVD $X = U\Sigma V^\top$

$$\mathrm{T}_{\mathcal{M}_s}(X) = \left\{ [U\ U_\perp] \begin{bmatrix} \mathbb{R}^{s\times s} & \mathbb{R}^{s\times(n-s)} \\ \mathbb{R}^{(m-s)\times s} & 0^{(m-s)\times(n-s)} \end{bmatrix} [V\ V_\perp]^\top \right\}$$

$$\mathrm{N}_{\mathcal{M}_s}(X) = \left\{ [U\ U_\perp] \begin{bmatrix} 0^{s\times s} & 0^{s\times(n-s)} \\ 0^{(m-s)\times s} & \mathbb{R}^{(m-s)\times(n-s)} \end{bmatrix} [V\ V_\perp]^\top \right\}$$

Geometry of $\mathcal{M}_{\le r}$ at $X = U\Sigma V^\top \in \mathcal{M}_s$ [Luke'13; Cason-Absil-Van Dooren'13; Schneider-Uschmajew'15]

$$\mathrm{T}_{\mathcal{M}_{\le r}}(X) = \mathrm{T}_{\mathcal{M}_s}(X) + \{R \in \mathrm{N}_X\mathcal{M}_s :\ \mathrm{rank}\,(R) \le r - s\}$$

$$\hat{\mathrm{N}}_{\mathcal{M}_{\le r}}(X) = \begin{cases} \mathrm{N}_{\mathcal{M}_s}(X), & \textit{if } s = r \\ \{0\}\,, & \textit{if } s < r \end{cases}$$

$$\mathrm{N}_{\mathcal{M}_{\le r}}(X) = \{Y \in \mathrm{N}_{\mathcal{M}_s}(X) : \mathrm{rank}(Y) \le \min\{m, n\} - r\}$$

All the cone mappings are not continuous at **singular** points [Olikier-Absil'22]

First- and second-order tangent sets

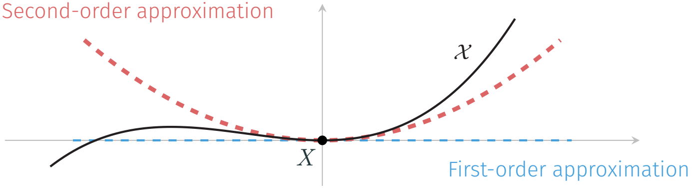
Motivation and challenge

First- and second-order tangent sets

$$\mathrm{T}_{\mathcal{X}}(X) = \{\eta \in \mathcal{E} : \exists t_i \to 0, \text{ s. t. } \operatorname{dist}(X + t_i\eta, \mathcal{X}) = o(t_i)\}$$

$$\mathrm{T}^2_{\mathcal{X}}(X, \eta) = \left\{\zeta \in \mathcal{E} : \exists t_i \to 0, \text{ s. t. } \operatorname{dist}(X + t_i\eta + \frac{1}{2}t_i^2\zeta, \mathcal{X}) = o(t_i^2)\right\}$$

Approximations to the set $\mathcal{X} = \{(x, y) \in \mathbb{R}^2 \mid y = x^2 + x^3\}$



Challenge when $\mathcal{X} = \mathcal{M}_{\leq r} = \left\{X \in \mathbb{R}^{m\times n} \mid \operatorname{rank}(X) \leq r\right\}$

- The geometry of $\mathcal{M}_{\leq r}$ is **complicated** [Olikier-Absil'22]
- The mapping $\operatorname{rank}(\cdot)$ is **not continuous**

Characterize low-rank sets via singular value functions

A new perspective

$$\mathcal{M}_{\le r} = \{X \in \mathbb{R}^{m\times n} \mid \sigma_{r+1}(X) = 0\}$$

- ☹ σ_{r+1} is both non-differentiable and non-convex
- ☺ σ_{r+1} is Lipschitz continuous [Weyl 1912]

$$|\sigma_{r+1}(X) - \sigma_{r+1}(X+\Delta)| \le \|\Delta\|_2$$

- ☺ σ_{r+1} controls the distance to $\mathcal{M}_{\le r}$

$$\mathrm{dist}(\tilde{X}, \mathcal{M}_{\le r}) \le (\min\{m,n\} - r)^{1/2}\sigma_{r+1}(\tilde{X})$$

Error bound condition at X

Let $\mathcal{M} := \{\tilde{X} \in \mathbb{R}^q \mid c_1(\tilde{X}) = 0, c_2(\tilde{X}) \le 0\}$. There exists a neighborhood $\mathcal{B}$ of $X \in \mathcal{M}$ and a constant $\rho > 0$ such that

$$\mathrm{dist}(\tilde{X}, \mathcal{M}) \le \rho\|(c_1(\tilde{X}),\ [c_2(\tilde{X})]_+)\|_2, \quad \text{for all } \tilde{X} \in \mathcal{B}$$

Bridge directional derivatives and tangent sets

Theorem

Let $\mathcal{M} = \{\tilde{X} \in \mathbb{R}^q \mid c_1(\tilde{X}) = 0, c_2(\tilde{X}) \leq 0\}$ with c_1 and c_2 satisfying

- error bound condition
- local Lipschitz property

Define the active index set $I_0(X) := \{j \in \{1, \dots, n_2\} \mid c_2(X)_j = 0\}$

(i) *(First-order)* If c_1 and c_2 are directionally differentiable at X, then

$$\mathrm{T}_{\mathcal{M}}(X) = \left\{\eta \in \mathbb{R}^q \mid c_1'(X;\eta) = 0,\ c_2'(X;\eta)_j \leq 0 \text{ for all } j \in I_0(X)\right\}$$

(ii) *(Second-order)* If, in addition, c_1 and c_2 admit parabolic second-order directional derivatives at X, then for any $\eta \in \mathrm{T}_{\mathcal{M}}(X)$

$$\mathrm{T}^2_{\mathcal{M}}(X;\eta) = \left\{\zeta \in \mathbb{R}^q \mid c_1''(X;\eta,\zeta) = 0,\ c_2''(X;\eta,\zeta)_j \leq 0 \text{ for all } j \in I_1(X;\eta)\right\}$$

where $I_1(X;\eta) := \{j \in I_0(X) \mid c_2'(X;\eta)_j = 0\}$

Application to low-rank matrices and tensors

Low-rank matrices

$$\mathcal{M}_{\leq r} = \{X \in \mathbb{R}^{m \times n} \mid \sigma_{r+1}(X) = 0\}$$

- $\| Y - \mathcal{P}_{\mathcal{M}_{\leq r}}(Y) \|_{\mathrm{F}} = \sqrt{\sum_{i=r+1}^{\min\{m,n\}} \sigma_i^2} \leq (\min\{m, n\} - r)^{1/2} \sigma_{r+1}$
- $\mathrm{T}_{\mathcal{M}_{\leq r}}(X) = \{\eta \in \mathbb{R}^{m \times n} \mid \sigma'_{r+1}(X; \eta) = 0\}$

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- $\mathrm{T}_{\mathcal{M}_{\le r}}(X) = \{\eta \in \mathbb{R}^{m\times n} \mid \sigma'_{r+1}(X;\eta) = 0\}$

Low-rank tensors hierarchical Tucker, Tucker, and tensor train

$$\mathcal{M}^{\mathrm{ht}}_{\le \mathbf{r}} = \bigcap_{t\in T} \mathrm{ten}^{\mathrm{ht}}_{(t)}\left(\mathbb{R}^{n_t\times n_{t\_}}_{\le r_t}\right) = \left\{\mathbf{X} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d} \mid \sigma_{r_t+1}(X^{\mathrm{ht}}_{(t)}) = 0 \text{ for } t \in T\right\}$$

- $\|\mathbf{Y} - \mathcal{P}^{\mathrm{HOSVD}}_{\le \mathbf{r}}(\mathbf{Y})\|_{\mathrm{F}} \le \rho' \sqrt{\sum_{t\in T} \sigma^2_{r_t+1}(Y^{\mathrm{ht}}_{(t)})}$
- $\mathrm{T}_{\mathcal{M}^{\mathrm{ht}}_{\le \mathbf{r}}}(\mathbf{X}) = \bigcap_{t\in T} \mathrm{ten}^{\mathrm{ht}}_{(t)}\left(\mathrm{T}_{\mathcal{R}_t}(X^{\mathrm{ht}}_{(t)})\right)$ with $\mathcal{R}_t := \mathbb{R}^{n_t\times n_{t\_}}_{\le r_t}$

Application to symmetric matrices

Low-rank symmetric matrices

$$\mathcal{S}_{\leq r}(n) = \{X \in \mathrm{Sym}(n) \mid \mathrm{rank}(X) \leq r\}$$

Application to symmetric matrices

Low-rank symmetric matrices

$$\mathcal{S}_{\leq r}(n) = \{X \in \mathrm{Sym}(n) \mid \mathrm{rank}(X) \leq r\} \; = \; \boxed{?}$$

- A decomposition

$$\mathcal{S}_{\leq r}(n) = \bigcup_{j=1}^{r+1} \mathcal{S}_j, \text{ with } \mathcal{S}_j := \{X \in \mathrm{Sym}(n) \mid \lambda_j(X) = 0,\ \lambda_{j+n-r-1}(X) = 0\}$$

- Verify the error bound condition of $(\lambda_j,\ \lambda_{j+n-r-1})$
- Derivation of tangent sets $\mathrm{T}_{\bigcup_{j=1}^{r+1} \mathcal{S}_j}(X) = \bigcup_{j=1}^{r+1} \mathrm{T}_{\mathcal{S}_j}(X)$

Application to symmetric matrices

Low-rank symmetric matrices

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Low-rank PSD matrices

$$\mathcal{S}^{+}_{\leq r}(n) = \{X \in \mathrm{Sym}(n) \mid \lambda_{r+1}(X) = 0, \lambda_n(X) = 0\}$$

Application to symmetric matrices

Low-rank symmetric matrices

$$\mathcal{S}_{\leq r}(n) = \{X \in \mathrm{Sym}(n) \mid \mathrm{rank}(X) \leq r\} \;=\; \boxed{?}$$

- A decomposition

$$\mathcal{S}_{\leq r}(n) = \bigcup_{j=1}^{r+1} \mathcal{S}_j, \text{ with } \mathcal{S}_j := \{X \in \mathrm{Sym}(n) \mid \lambda_j(X) = 0,\ \lambda_{j+n-r-1}(X) = 0\}$$

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Low-rank PSD matrices

$$\mathcal{S}^+_{\leq r}(n) = \{X \in \mathrm{Sym}(n) \mid \lambda_{r+1}(X) = 0, \lambda_n(X) = 0\}$$

$\mathcal{S}^+_{\leq r}(n)$ coincides with $\mathcal{S}_{r+1}$

Directional derivatives of singular values

Necessary notation [Zhang-Zhang-Xiao'13 SVVA]

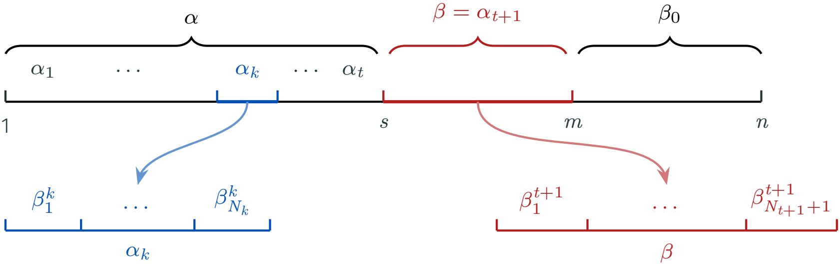
- the SVD $X = \bar{U}[\bar{\Sigma}(X)\ 0]\,\bar{V}^\top$ with $\sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_m(X)$

$$\alpha = \{i : \sigma_i(X) > 0, 1 \leq i \leq m\}, \quad \beta = \{i : \sigma_i(X) = 0, 1 \leq i \leq m\}$$

$$\beta_0 = \{m+1, \ldots, n\}$$

- the distinct singular values $\mu_1 > \mu_2 > \ldots > \mu_p > \mu_{p+1} = 0$

$$\alpha_k = \{i : \sigma_i(X) = \mu_k, 1 \leq i \leq m\}, \ k = 1, \ldots, p, \text{ and } \widehat{\beta} = \beta \cup \beta_0$$

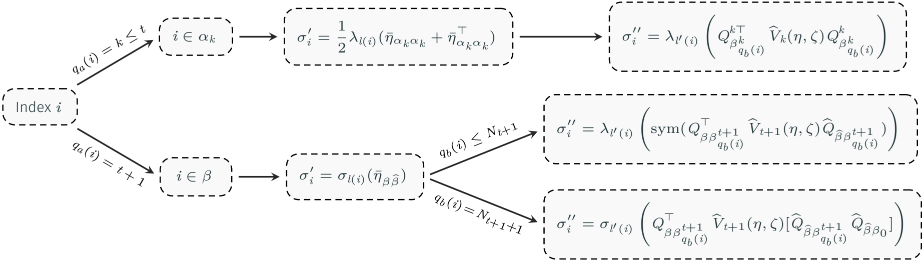


Directional derivatives of singular values (cont'd)

Four mappings

$$
\begin{aligned}
q_a &: \{1,\ldots,m\} \to \{1,\ldots,p+1\},\, q_a(i) = k, \text{ if } i \in \alpha_k \\
l &: \{1,\ldots,m\} \to \mathbb{N},\, l(i) = i - \kappa_{q_a(i)-1} \\
q_b &: \{1,\ldots,m\} \to \mathbb{N},\, q_b(i) = e, \text{ if } l(i) \in \beta_e^{q_a(i)} \\
l' &: \{1,\ldots,m\} \to \mathbb{N},\, l'(i) = l(i) - \kappa^{(q_a(i))}_{q_b(i)-1}
\end{aligned}
$$

Formula of $\sigma_i'(X;\eta)$ and $\sigma_i''(X;\eta,\zeta)$ at $X = \bar{U}[\bar{\Sigma}(X)\ 0]\,\bar{V}^\top$ for $i = 1, 2, \ldots, m$



Index i

$q_a(i) = k \le t$

$i \in \alpha_k$

$\sigma_i' = \frac{1}{2}\lambda_{l(i)}(\bar{\eta}_{\alpha_k\alpha_k} + \bar{\eta}_{\alpha_k\alpha_k}^\top)$

$\sigma_i'' = \lambda_{l'(i)}\left(Q^{k\top}_{\beta^k_{q_b(i)}}\widehat{V}_k(\eta,\zeta)\,Q^k_{\beta^k_{q_b(i)}}\right)$

$q_a(i) = t+1$

$i \in \beta$

$\sigma_i' = \sigma_{l(i)}(\bar{\eta}_{\beta\widehat{\beta}})$

$q_b(i) \le N_{t+1}$

$\sigma_i'' = \lambda_{l'(i)}\left(\mathrm{sym}(Q^\top_{\beta\beta^{t+1}_{q_b(i)}}\widehat{V}_{t+1}(\eta,\zeta)\widehat{Q}_{\widehat{\beta}\beta^{t+1}_{q_b(i)}})\right)$

$q_b(i) = N_{t+1}+1$

$\sigma_i'' = \sigma_{l'(i)}\left(Q^\top_{\beta\beta^{t+1}_{q_b(i)}}\widehat{V}_{t+1}(\eta,\zeta)[\widehat{Q}_{\widehat{\beta}\beta^{t+1}_{q_b(i)}}\ \widehat{Q}_{\widehat{\beta}\beta_0}]\right)$

Compute second-order tangent set

Important geometry of $\mathcal{M}_{\leq r}$

- consider $X = \bar{U}[\Sigma\ 0]\bar{V}^\top \in \mathcal{M}_s$ with $\bar{U} = [U\ U_\perp]$ and $\bar{V} = [V\ V_\perp]$
- $\eta \in \mathrm{T}_{\mathcal{M}_{\leq r}}(X)$ if and only if

$$\eta = UAV^\top + U_\perp BV^\top + UCV_\perp^\top + U_\perp RV_\perp^\top$$

where $\operatorname{rank}(R) = \ell - s \leq r - s$

Index mappings associated with $\sigma_{r+1}(X) = 0$

- $r+1 \in \beta$, $q_a(r+1) = p+1$, $l(r+1) = r+1-s$
- $\bar{\eta}_{\beta\widehat{\beta}} = U_\perp^\top \eta V_\perp = R$ and $\operatorname{rank}(R) = \ell - s \Rightarrow q_b(r+1) = N_{p+1}+1$
- $l'(r+1) = \ell + 1 - s$

$$\begin{tikzcd}
 & & \sigma_i'' = \lambda_{l'(i)}(\operatorname{sym}(Q^T_{\beta\beta^{p+1}_{q_b(i)}}\widehat{V}_{p+1}(\eta,\zeta)\widehat{Q}_{\widehat{\beta}\beta^{p+1}_{q_b(i)}})) \\
i \in \beta \arrow[r] & \sigma_i' = \sigma_{l(i)}(\bar{\eta}_{\beta\widehat{\beta}}) \arrow[ur, "q_b(i) \leq N_{p+1}"] \arrow[dr, "q_b(i) = N_{p+1+1}"'] & \\
 & & \sigma_i'' = \sigma_{l'(i)}(Q^T_{\beta\beta^{p+1}_{q_b(i)}}\widehat{V}_{p+1}(\eta,\zeta)[\widehat{Q}_{\widehat{\beta}\beta^{p+1}_{q_b(i)}}\ \widehat{Q}_{\widehat{\beta}\beta_0}])
\end{tikzcd}$$

Second-order tangent set to $\mathcal{M}_{\leq r}$

Theorem

Given $X \in \mathcal{M}_{\leq r}$ and $\eta \in \mathrm{T}_{\mathcal{M}_{\leq r}}(X)$ with $\operatorname{rank}(X) = s$ and the SVD $X = U\Sigma V^\top$. Let $\mathcal{P}_{\mathrm{N}_{\mathcal{M}_s}(X)}(\eta) = U_\eta \Sigma_\eta V_\eta^\top$, with $\operatorname{rank}(\Sigma_\eta) = \ell - s$. Take $[U\ U_\eta\ \ U_{\eta\perp}] \in \mathcal{O}(m)$ and $[V\ V_\eta\ \ V_{\eta\perp}] \in \mathcal{O}(n)$. It holds that

$$\mathrm{T}^2_{\mathcal{M}_{\leq r}}(X;\eta) = \left\{ 2\eta X^\dagger \eta + [U^+\ \ U_{\eta\perp}] \begin{bmatrix} W_1 & W_2 \\ W_3 & J \end{bmatrix} [V^+\ \ V_{\eta\perp}]^\top \;\middle|\; \begin{array}{l} W_1 \in \mathbb{R}^{\ell\times\ell} \\ W_2 \in \mathbb{R}^{\ell\times(n-\ell)} \\ W_3 \in \mathbb{R}^{(m-\ell)\times\ell} \\ J \in \mathbb{R}^{(m-\ell)\times(n-\ell)} \\ \operatorname{rank}(J) \leq r-\ell \end{array} \right\}$$

where $U^+ = [U\ U_\eta]$, $V^+ = [V\ V_\eta]$

A unified analysis of tangent sets to low-rank sets

Set	Format	First-order	Second-order
$\mathcal{M}$	general	Theorem 1	Theorem 1
$\mathcal{M}_{\leq r}$	matrix	[Schneider-Uschmajew'15 SIOPT]	Proposition 1
$\mathcal{M}^{\mathrm{ht}}_{\leq \mathbf{r}}$	hierarchical Tucker	Proposition 2	Proposition 2
$\mathcal{M}^{\mathrm{tc}}_{\leq \mathbf{r}}$	Tucker	[Gao-Peng-Yuan'25 MP]	Proposition 2
$\mathcal{M}^{\mathrm{tt}}_{\leq \mathbf{r}}$	tensor train	[Kutschan'18 LAA]	Proposition 2
$\mathcal{S}_{\leq r}(n)$	symmetric matrix	[Li-Xiu-Zhou'20 JOTA]	Proposition 3
$\mathcal{S}^{+}_{\leq r}(n)$	PSD matrix	[Levin-Kileel-Boumal'25 MP]	Proposition 4

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Extension to intersection of sets $\mathcal{M} \cap \mathcal{K}$?

Intersection of sets

Theorem

- $\mathcal{M} = \{\tilde{X} \in \mathbb{R}^q \mid c(\tilde{X}) = 0\}$ with $c : \mathbb{R}^q \to \mathbb{R}^{n_1}$,
- $\mathcal{K} = \{\tilde{X} \in \mathbb{R}^q \mid h(\tilde{X}) = 0\}$ with smooth $h : \mathbb{R}^q \to \mathbb{R}^k$

Intersection of sets

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- $\mathcal{M} = \{\tilde{X} \in \mathbb{R}^q \mid c(\tilde{X}) = 0\}$ with $c : \mathbb{R}^q \to \mathbb{R}^{n_1}$,
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- Suppose that $(\mathcal{M}, \mathcal{K})$ satisfy

$$\begin{tikzcd}
\text{(transversal)}\ \overline{\mathcal{M}} \cap \overline{\mathcal{K}} \subseteq \mathbb{R}^p \arrow[d, "\text{(open) } \phi"'] \arrow[dr, "h \circ \phi\ \text{(rank-constant)}"] & \\
\mathcal{M} \cap \mathcal{K} \subseteq \mathbb{R}^q \arrow[r, "h"'] & \mathbb{R}^k
\end{tikzcd}$$

Intersection of sets

Theorem

- $\mathcal{M} = \{\tilde{X} \in \mathbb{R}^q \mid c(\tilde{X}) = 0\}$ with $c : \mathbb{R}^q \to \mathbb{R}^{n_1}$,
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\mathcal{M} \cap \mathcal{K} \subseteq \mathbb{R}^q \arrow[r, "h"'] & \mathbb{R}^k
\end{tikzcd}$$

We have the following intersection rules for the tangent sets

$$\mathrm{T}_{\mathcal{M}\cap\mathcal{K}}(X) = \mathrm{T}_{\mathcal{M}}(X) \cap \mathrm{T}_{\mathcal{K}}(X)$$

$$\mathrm{T}^2_{\mathcal{M}\cap\mathcal{K}}(X;\eta) = \mathrm{T}^2_{\mathcal{M}}(X;\eta) \cap \mathrm{T}^2_{\mathcal{K}}(X;\eta)$$

Applications

Low-rank matrix sets $\mathcal{M}_{\le r} \cap \mathcal{H}$

- $\mathcal{H} = \mathrm{Aff}(m,n) = \{X \in \mathbb{R}^{m\times n} \mid \mathcal{A}(X) - b = 0\}$ [Li-Luo'23 SIOPT]
- $\mathcal{H} = \mathrm{S}_{\mathrm{F}}(m,n) = \{X \in \mathbb{R}^{m\times n} \mid \|X\|_{\mathrm{F}}^2 - 1 = 0\}$ [Cason-Absil-Van Dooren'13 LAA]
- $\mathcal{H} = \mathrm{Ob}(m,n) = \{X \in \mathbb{R}^{m\times n} \mid \mathrm{diag}(XX^\top) - \mathbf{1} = \mathbf{0}\}$ [Yang-Gao-Yuan'25]
- $\mathcal{H} = \mathbb{H}^m_{n-1} = \{X \in \mathbb{R}^{m\times n} \mid \mathrm{diag}(X\,\mathrm{Diag}(-1,1,\ldots,1)X^\top) + \mathbf{1} = \mathbf{0}\}$ ☺ new

A smooth parameterization

$$(\mathcal{M}_{\mathrm{LR}}, \phi_{\mathrm{LR}}) = (\mathbb{R}^{m\times r} \times \mathbb{R}^{n\times r},\ (L,R) \mapsto LR^\top)$$

☺ $\mathcal{M}_{\mathrm{LR}}$ coincides with the ambient space

☺ ϕ_{LR} is open around "balanced" (L,R)

★ It remains to prove $\overline{\mathcal{H}} = \phi_{\mathrm{LR}}^{-1}(\mathcal{H})$ is manifold

$$\begin{tikzcd}
\text{(transversal)}\ \mathcal{M}_{\mathrm{LR}} \cap \overline{\mathcal{H}} \arrow[d, "\text{(open)}\ \phi_{\mathrm{LR}}"'] \arrow[dr, "h \circ \phi\ \text{(rank-constant)}"] & \\
\mathcal{M}_{\le r} \cap \mathcal{H} \arrow[r, "h"'] & \mathbb{R}^k
\end{tikzcd}$$

A unified analysis of tangent sets

Set	Format	First-order	Second-order
$\mathcal{M}$	Assumption 1	Theorem 1	Theorem 1
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$\mathcal{S}^{+}_{\leq r}(n)$	PSD matrix	[Levin-Kileel-Boumal'25 MP]	Proposition 4
Intersection of sets	**Structured set**	**First-order**	**Second-order**
$\mathcal{M} \cap \mathcal{K}$	Assumption 2	Theorem 2	Theorem 2
$\mathcal{M}_{\leq r} \cap \mathcal{H}$	$\mathcal{H}$ is an affine manifold	[Li-Luo'23 SIOPT]	Appendix C.1
$\mathcal{M}_{\leq r} \cap \mathcal{H}$	$\mathcal{H}$ is orthogonally invariant	[Yang-Gao-Yuan'25]	Appendix C.2
$\mathcal{M}_{\leq r} \cap \mathcal{H}$	$\mathcal{H}$ is hyperbolic	Appendix C.3	Appendix C.3
$\mathcal{S}_{\leq r}(n) \cap \mathcal{U}$	$\mathcal{U} = \{X \mid \|X\|_{\mathrm{F}}^2 = 1\}$	Appendix D.2	Appendix D.2
$\mathcal{S}^{+}_{\leq r}(n) \cap \mathcal{U}$	$\mathcal{U} = \{X \mid \mathcal{A}(X) = b\}$	[Levin-Kileel-Boumal'25 MP]	Appendix D.3

Tangent sets bridge optimization landscapes

Second-order optimality conditions

Second-order stationarity [Ruszczynski'06]

$$\min_{X \in \mathcal{M}} f(X),$$

- $-\nabla f(X^*) \in \hat{\mathrm{N}}_{\mathcal{M}}(X^*)$
- $\langle \nabla f(X^*), \zeta \rangle + \big\langle \eta, \nabla^2 f(X^*)[\eta] \big\rangle \geq 0$, for every $\eta \in \mathrm{T}_{\mathcal{M}}(X)$ such that $\langle \nabla f(X^*), \eta \rangle = 0$ and all $\zeta \in \mathrm{T}^2_{\mathcal{M}}(X^*; \eta)$

Smooth parameterization finds first-order stationary points [Levin-Kileel-Boumal'25 MP]

$$\begin{tikzcd}
\overline{\mathcal{M}} \subseteq \mathcal{W} \arrow[d, "\phi"'] \arrow[dr, "\bar{f} = f \circ \phi"] & \\
\mathcal{M} \subseteq \mathcal{E} \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

$$\min_{Y \in \overline{\mathcal{M}}} \bar{f}(Y) = f(\phi(Y))$$

$$\min_{X \in \mathcal{M}} f(X)$$

When does a parameterization satisfy "$2 \Rightarrow 2$"?

A sufficient and necessary condition for "$2 \Rightarrow 2$"

Mappings connecting the geometry

$$\mathbf{L}_Y : \mathrm{T}_{\overline{\mathcal{M}}}(Y) \to \mathcal{E} : \; v \mapsto \mathrm{D}\phi_Y[v], \quad \text{for } Y \in \overline{\mathcal{M}}$$

$$\mathbf{Q}_{Y,v} : \mathrm{T}^2_{\overline{\mathcal{M}}}(Y; v) \to \mathcal{E} : \; u_v \mapsto \mathrm{D}\phi_Y[u_v] + \mathrm{D}^2\phi_Y[v, v], \quad \text{for } Y \in \overline{\mathcal{M}} \text{ and } v \in \mathrm{T}_{\overline{\mathcal{M}}}(Y)$$

Two important sets

$$\mathcal{Q}_Y(v) := \bigcup_{\{v_i\}_{i\in\mathbb{N}} :\, \mathbf{L}_Y(v_i) \to \mathbf{L}_Y(v)} \lim_{i\to\infty} \left(\mathbf{Q}_{Y,v_i}(u_{v_i}) + \mathrm{im}(\mathbf{L}_Y)\right), \quad \text{for } v \in \mathrm{T}_{\overline{\mathcal{M}}}(Y)$$

$$\mathit{\Gamma}_Y := \left\{\mathcal{P}_{\hat{\mathrm{N}}_{\mathcal{M}}(\phi(Y))} \left(\mathrm{D}\phi_Y[\mathrm{II}(v_0, v_1)] + \mathrm{D}^2\phi_Y[v_0, v_1]\right) \mid v_0 \in \ker(\mathbf{L}_Y),\; v_1 \in \mathrm{T}_{\overline{\mathcal{M}}}(Y)\right\}$$

Theorem

The parameterization $(\overline{\mathcal{M}}, \phi)$ satisfies "$2 \Rightarrow 2$" at $Y \in \overline{\mathcal{M}}$ if and only if $\mathrm{im}(\mathbf{L}_Y) = \mathrm{T}_{\mathcal{M}}(X)$ where $X = \phi(Y)$, and for all $v \in \mathrm{T}_{\overline{\mathcal{M}}}(Y)$

$$\mathrm{T}^2_{\mathcal{M}}(X; \mathbf{L}_Y(v)) \subseteq \overline{\mathrm{conv}}\left(\mathcal{Q}_Y(v) + \mathcal{Q}_Y(0) + \mathit{\Gamma}_Y\right)$$

Second-order optimality on bounded-rank matrices

Second-order stationarity on low-rank optimization

Proposition

$$\begin{aligned} \min_{X\in\mathbb{R}^{m\times n}} \quad & f(X) \\ \mathrm{s.\,t.} \quad & X \in \mathcal{M}_{\leq r} \cap \mathcal{H} \end{aligned}$$

The point X^* is second-order stationary if

$$\langle \nabla f(X^*), \eta\rangle = 0, \text{ for all } \eta \in \mathrm{T}_{\mathcal{M}_{\leq r}}(X^*) \cap \mathrm{T}_{\mathcal{H}}(X^*)$$

and for all $\eta \in \mathrm{T}_{\mathcal{M}_{\leq r}}(X^*) \cap \mathrm{T}_{\mathcal{H}}(X^*)$, it holds that

$$\langle \nabla f(X^*), \zeta\rangle + \left\langle \eta, \nabla^2 f(X^*)[\eta]\right\rangle \geq 0, \quad \text{for all } \zeta \in \mathrm{T}^2_{\mathcal{M}_{\leq r}}(X^*;\eta) \cap \mathrm{T}^2_{\mathcal{H}}(X^*;\eta)$$

Corollary $\mathcal{H} = \mathbb{R}^{m\times n}$

$$\begin{aligned} \min_{X\in\mathbb{R}^{m\times n}} \quad & f(X) \\ \mathrm{s.\,t.} \quad & X \in \mathcal{M}_{\leq r} \end{aligned}$$

The point $X^* \in \mathcal{M}_{\leq r}$ with $\mathrm{rank}(X^*) = s$ is second-order stationary if

$$\begin{cases} \nabla_{\mathcal{M}_r} f(X^*) = 0 \text{ and } \nabla^2_{\mathcal{M}_r} f(X^*) \succeq 0, & \text{if } s = r \\ \nabla f(X^*) = 0 \text{ and } \left\langle \eta, \nabla^2 f(X^*)[\eta]\right\rangle \geq 0 \text{ for all } \eta \in \mathrm{T}_{\mathcal{M}_{\leq r}}(X^*), & \text{if } s < r \end{cases}$$

Complexity of verifying second-order conditions (SOC)

Find a negative curvature direction at X with $\operatorname{rank}(X) < r$

$$\begin{aligned} \min_{\eta \in \mathbb{R}^{m \times n}} \quad & \langle \eta, \mathcal{A}(\eta) \rangle \\ \text{s.t.} \quad & \|\eta\|_{\mathrm{F}} = 1, \\ & \eta \in \mathrm{T}_{\mathcal{M}_{\le r}}(X). \end{aligned} \tag{Q}$$

Problem: VERSOC
Input: Parameters m, n, r; point $X \in \mathcal{M}_{\le r}$; symmetric operator $\mathcal{A}$
Question: Does the optimal value of (Q) $\lambda^\star < 0$?

Complexity of verifying second-order conditions (SOC)

Find a negative curvature direction at X with $\mathrm{rank}(X) < r$

$$\begin{aligned} \min_{\eta \in \mathbb{R}^{m \times n}} \quad & \langle \eta, \mathcal{A}(\eta) \rangle \\ \text{s. t.} \quad & \|\eta\|_{\mathrm{F}} = 1, \\ & \eta \in \mathrm{T}_{\mathcal{M}_{\leq r}}(X). \end{aligned} \tag{Q}$$

Problem: VERSOC
Input: Parameters m, n, r; point $X \in \mathcal{M}_{\leq r}$; symmetric operator $\mathcal{A}$
Question: Does the optimal value of (Q) $\lambda^\star < 0$?

An NP-complete problem [Karp'72]

Problem: CLIQUE
Input: Undirected graph $G = (\mathcal{V}, E)$; clique size K
Question: Does there exist a clique of size K in G?

NP-hardness of verifying SOC

Theorem (NP-hardness)

The problem CLIQUE is polynomially reducible to VERSOC, and thus verifying second-order optimality conditions is NP-hard.

NP-hardness of verifying SOC

Theorem (NP-hardness)

The problem CLIQUE is polynomially reducible to VERSOC, and thus verifying second-order optimality conditions is NP-hard.

Theorem (No FPTAS)

Unless P=NP, there is no fully polynomial-time approximation scheme for verifying whether a point is second-order stationary for optimization problems over bounded-rank matrices.

Equivalence of problems

$$\begin{tikzcd}
\overline{\mathcal{M}} \arrow[d, "\phi"'] \arrow[dr, "\bar{f} = f \circ \phi"] & \\
\mathcal{M}_{\leq r} \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

$$\min_{Y \in \overline{\mathcal{M}}} \quad \bar{f}(Y) = f(\phi(Y))$$

$$\min_{X \in \mathcal{M}_{\leq r}} \quad f(X)$$

Smooth parameterization of $\mathcal{M}_{\leq r}$

- LR factorization: $\overline{\mathcal{M}} = \mathbb{R}^{m \times r} \times \mathbb{R}^{n \times r}, \ \phi(L, R) = LR^\top$
- Desingularization: $\overline{\mathcal{M}} = \left\{(X, G) \in \mathbb{R}^{m \times n} \times \mathrm{Gr}(n, n-r) \mid XG = 0\right\}, \ \phi(X, G) = X$

$\mathrm{T}_{\mathcal{M}_{\leq r}}(X) = \mathrm{im}(\mathbf{L}_Y)$ and $\mathrm{T}^2_{\mathcal{M}_{\leq r}}(X; \mathbf{L}_Y(v)) = \mathrm{im}(\mathbf{Q}_{Y,v})$ when $\mathrm{rank}(X) = r$

Proposition

The parameterizations of $\mathcal{M}_{\leq r}$ satisfy "$2 \Rightarrow 2$" if and only if $\mathrm{rank}(X) = r$.

Conclusion and perspective

A unified analysis to derive tangent sets

★ Second-order optimality conditions [Bonnans-Cominetti-Shapiro'99 SIOPT]

- Metric subregularity [Gfrerer'11 SIOPT]
- System stability [Gfrerer-Mordukhovich'17 SIOPT]

Future work

☺ Efficient algorithm exploiting the geometry

$$\mathrm{T}_{\mathcal{H}\cap\mathcal{M}}(X) = \mathrm{T}_{\mathcal{H}}(X) \cap \mathrm{T}_{\mathcal{M}}(X)$$

? Second-order development beneficial from the property

$$\mathrm{T}^2_{\mathcal{H}\cap\mathcal{M}}(X;\eta) = \mathrm{T}^2_{\mathcal{H}}(X;\eta) \cap \mathrm{T}^2_{\mathcal{M}}(X;\eta)$$

Conclusion and perspective

A unified analysis to derive tangent sets

★ Second-order optimality conditions [Bonnans-Cominetti-Shapiro'99 SIOPT]

- Metric subregularity [Gfrerer'11 SIOPT]
- System stability [Gfrerer-Mordukhovich'17 SIOPT]

Future work

☺ Efficient algorithm exploiting the geometry

$$\mathrm{T}_{\mathcal{H}\cap\mathcal{M}}(X)=\mathrm{T}_{\mathcal{H}}(X)\cap\mathrm{T}_{\mathcal{M}}(X)$$

? Second-order development beneficial from the property

$$\mathrm{T}^2_{\mathcal{H}\cap\mathcal{M}}(X;\eta)=\mathrm{T}^2_{\mathcal{H}}(X;\eta)\cap\mathrm{T}^2_{\mathcal{M}}(X;\eta)$$

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Thanks for your attention!

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