A Space-decoupling Framework for Optimization on Bounded-rank Matrices with Orthogonally Invariant Constraints

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Joint work with **Dr. Bin Gao** and **Prof. Ya-xiang Yuan**

Outline

Low-rank optimization

Low-rank problems: a geometric view

Optimization on the fixed-rank manifold

$$\min \; f(X) \quad \text{s.t. } X \in \mathbb{R}_r^{m\times n} := \{X \in \mathbb{R}^{m\times n} : \operatorname{rank}(X) = r\}$$

- $\mathbb{R}_r^{m\times n}$ is a smooth manifold [Helmke and Shayman, 1995]
- Riemannian optimization algorithms [Vandereycken, 2013]
- $\mathbb{R}_r^{m\times n}$ is not closed

Optimization on the set of bounded-rank matrices

$$\min \; f(X) \quad \text{s.t. } X \in \mathbb{R}_{\le r}^{m\times n} := \{X \in \mathbb{R}^{m\times n} : \operatorname{rank}(X) \le r\}$$

- $\mathbb{R}_{\le r}^{m\times n}$ is a real-algebraic variety [Harris, 1992]
- Geometric methods [Schneider and Uschmajew, 2015; Hosseini and Uschmajew, 2019; Gao and Absil, 2022; Olikier et al., 2022; Levin et al., 2023; Rebjock and Boumal, 2024]
- $\mathbb{R}_{\le r}^{m\times n}$ is closed

Bounded-rank matrices + orthogonally invariant constraints

$$\begin{aligned} \min \quad & f(X) \\ \text{s.t.} \quad & \operatorname{rank}(X) \leq r \\ & h(X) = 0 \end{aligned}$$

Low rank + additional constraints

Bounded-rank matrices + orthogonally invariant constraints

$$\begin{array}{ll} \min & f(X) \\ \text{s.t.} & \operatorname{rank}(X) \leq r \\ & h(X) = 0 \end{array}$$

Blanket assumptions

- Objective: $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is twice continuously differentiable
- Smooth and orthogonally invariant constraints

$$h(XQ) = h(X), \text{ for } Q \in \mathcal{O}(n)$$

- The differential $\mathrm{D}h$ has full rank in

$$\mathcal{H} := \left\{X \in \mathbb{R}^{m\times n} : \ h(X) = 0\right\}$$

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$\mathcal{H}$ is a smooth manifold embedded in $\mathbb{R}^{m\times n}$

Applications

Coupled feasible region

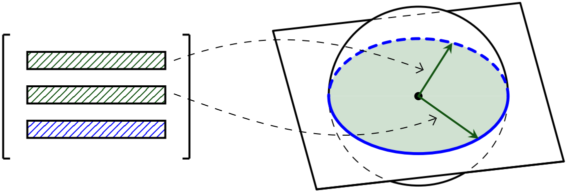
$$\mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H} \;\text{ with }\; \mathcal{H} = \{X :\ h(X) = 0\}$$

Low rank + oblique manifold

- Low-rank data fitting on sphere [Chu et al., 2005 SIMAX]

$$\mathcal{H} = \mathrm{Ob}(m, n) := \{X \in \mathbb{R}^{m\times n} :\ \mathrm{diag}\left(XX^\top\right) - \mathbf{1} = \mathbf{0}\}$$

$$\mathbb{R}^{3\times 4}_{\leq 2} \cap \mathrm{Ob}(3,4) \ni$$



Applications (cont'd)

Low rank + Frobenius sphere

- Low-rank approximation of graph similarity matrices [Cason et al., 2013 LAA]

$$\mathcal{H} = \mathrm{S}_{\mathrm{F}}(m, n) := \{X \in \mathbb{R}^{m \times n} :\ \|X\|_{\mathrm{F}}^2 - 1 = 0\}$$

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Low rank + stacked Stiefel manifold

- Low-rank solutions of synchronization problems [Boumal, 2015]

$$\mathcal{H} = \left\{X = (X_1; X_2; \ldots; X_k) \in \mathbb{R}^{kp\times n} :\ X_j^\top \in \mathrm{St}(n, p) \text{ for } j = 1, 2, \ldots, k\right\}$$

$$\begin{bmatrix} \boxed{X_1 :\ X_1 X_1^\top = I} \\ \boxed{X_2 :\ X_2 X_2^\top = I} \\ \boxed{X_3 :\ X_3 X_3^\top = I} \end{bmatrix} \in \mathcal{H}$$

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Low rank + SDPs

- Low-rank semidefinite programs [Journée et al., 2010; Tang and Toh, 2023]

$$\min \ \langle C, Y\rangle \qquad \mathrm{s.t.}\ \ Y \in \mathbb{R}^{n\times n}_{\le r} \cap \mathbb{S}^n_+,\ \ \mathcal{A}(Y) = b$$

$$\min \ \left\langle C, XX^\top \right\rangle \quad \mathrm{s.t.}\ \ X \in \mathbb{R}^{n\times n}_{\le r},\ \ \mathcal{A}(XX^\top) = b$$

Challenges

Constraint-coupled optimization

$$\begin{aligned} \min_{X \in \mathbb{R}^{m\times n}} \quad & f(X) \\ \text{s.t.} \quad & X \in \mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H} \end{aligned}$$

Challenges

- Unknown **optimality conditions**

  Bouligand-tangent cone of the coupled feasible region

- Inherently **non-smooth structure** of $\mathbb{R}^{m\times n}_{\leq r}$ [Levin et al., 2023 MP]
- Unclear **strategy to preserve feasibility** on $\mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H}$

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How to tackle the coupled and non-smooth constraints?

Bounded-rank optimization: optimality analysis

Variational geometry of $\mathbb{R}_{\leq r}^{m \times n}$

- Mordukhovich normal cone [Luke, 2013]
- Bouligand tangent cone and Frèchet normal cone [Schneider and Uschmajew, 2015]
- Clarke tangent cone and the corresponding normal cone [Hosseini et al., 2019; Li et al., 2019]

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Variational geometry of $\mathbb{R}^{m\times n}_{\le r} \cap \mathcal{C}$

- Mordukhovich normal cone when $\mathcal{C} = \mathrm{Sym}(n)$ [Tam, 2017]
- Frèchet normal cone when $\mathcal{C} = \mathrm{Sym}(n) \cap \mathcal{B}_i$ [Li et al., 2020]

  $\mathcal{B}_1 = \{X : \|X\|_\mathrm{F} \le 1\}$, $\mathcal{B}_2 = \{X : -tI_n \preceq X \preceq tI_n\}$, $\mathcal{B}_3 = \{X : X \succeq 0,\ \mathrm{Tr}(X) = 1\}$
- When $\mathcal{C}$ is a spectral set [Lewis, 1996; Pan, 2017; Li and Luo, 2023]

  $m \neq n$ **breaks the symmetry!**
- Bouligand tangent cone when $\mathcal{C} = \{X \in \mathbb{R}^{m\times n} : \|X\|_\mathrm{F} = 1\}$ [Cason et al., 2013]
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Not available for $\mathbb{R}^{m\times n}_{\le r} \cap \mathcal{H}$

Bounded-rank optimization: algorithms

- Projected gradient descent framework [Jain et al., 2014; Schneider and Uschmajew, 2015; Olikier et al., 2022; 2024]
- Retraction-free methods [Schneider and Uschmajew, 2015; Olikier et al., 2023; 2024]
- Optimizing over a smooth parameterization

$$\begin{tikzcd} \mathcal{M} \arrow[d, "\phi"'] \arrow[dr, "\bar{f} := f \circ \phi"] & \\ \mathbb{R}^{m \times n}_{\leq r} \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

$$\min \bar{f}(Y) \;\text{ s.t. }\; Y \in \mathcal{M}$$

- LR factorization [Mishra et al. 2012; Levin et al. 2022]

$$\mathcal{M} = \mathbb{R}^{m \times r} \times \mathbb{R}^{n \times r}, \;\phi(L, R) = LR^\top$$

- SVD-type lift [Mishra et al., 2014; Levin et al., 2023]

$$\mathcal{M} = \mathrm{St}(m, r) \times \mathrm{Sym}(r) \times \mathrm{St}(n, r), \;\phi(U, S, V) = USV^\top$$

- Desingularization [Khrulkov and Oseledets, 2018; Rebjock and Boumal, 2024]

$$\mathcal{M} = \left\{(X, G) \in \mathbb{R}^{m \times n} \times \mathrm{Gr}(n, n - r) : XG = 0\right\}, \;\phi(X, G) = X$$

$$\begin{aligned} \min_{X \in \mathbb{R}^{m\times n}} \quad & f(X) \\ \mathrm{s.\,t.} \quad & X \in \mathbb{R}^{m\times n}_{\le r} \cap \mathcal{H} \end{aligned}$$

How to tackle the coupled and non-smooth constraints?

A space-decoupling framework?

$$
\begin{aligned}
\min_{X \in \mathbb{R}^{m\times n}} \quad & f(X) \\
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Goal

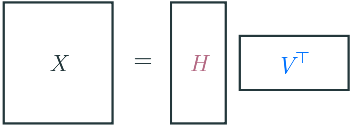
- optimality analysis
- smooth parametrization
- equivalence of two problems
- convergence analysis

Orthogonally invariant functions and optimality analysis

Orthogonal invariance “meets” low rank

Low-rank decomposition

- $\mathcal{H} = \{X \in \mathbb{R}^{m\times n} :\ h(X) = 0\}$ with orthogonally invariant h
- $X \in \mathbb{R}_s^{m\times n} \cap \mathcal{H}$ extracts low-rank (H) and orthogonally invariant (V)



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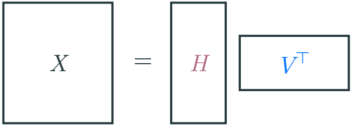

Submanifold inherites orthogonal invariance

$$X = HV^\top, \quad V^\top V = I$$

- natural embeddings $i^s(\cdot) : \ \mathbb{R}^{m\times s} \longrightarrow \mathbb{R}^{m\times n}, \ H \longmapsto [H \, 0^{m\times(n-s)}]$
- $h^s := h \circ i^s$ is orthogonally invariant
- $\mathcal{H}^s := \{H \in \mathbb{R}^{m\times s} : h^s(H) = 0\}$ is an embedded submanifold of $\mathbb{R}^{m\times s}$

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$$X = H \; V^\top$$

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$$\boxed{X \in \mathcal{H} \iff H \in \mathcal{H}^s}$$

Space decoupling - level 1: tangent space

$$\mathcal{H} = \{X \in \mathbb{R}^{m\times n} :\ h(X) = 0\} \quad \text{and} \quad h(XQ) = h(X) \text{ for } Q \in \mathcal{O}(n)$$

Definition

- *Bouligand tangent cone*:

$$\mathrm{T}_X\mathcal{X} := \left\{\eta = \lim_{k\to\infty} \frac{(X_k - X)}{t_k} :\ X_k \in \mathcal{X}, t_k > 0 \textit{ for all } k, \lim_{k\to\infty} t_k = 0\right\}.$$

- *Fréchet normal cone*: $\mathrm{N}_X\mathcal{X} := (\mathrm{T}_X\mathcal{X})^\circ$

Tangent space decoupling at $X = HV^\top \in \mathcal{H}$

$$\begin{aligned}\mathrm{T}_X\mathcal{H} &= \left\{KV^\top + BV_\perp^\top :\ K \in \mathrm{T}_H\mathcal{H}^s,\ B \in \mathbb{R}^{m\times(n-s)}\right\}\\ &= (\mathrm{T}_H\mathcal{H}^s)\ V^\top \oplus \left(\mathbb{R}^{m\times(n-s)}\right)\ V_\perp^\top\end{aligned}$$

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$$\mathrm{T}_X\left(\mathbb{R}^{m\times n}_{\le r} \cap \mathcal{H}\right)?$$

Smooth layers

$$\mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H} = \bigcup_{s=0}^{s=r} (\mathbb{R}^{m\times n}_{s} \cap \mathcal{H}):\;\; X = HV^\top = U\Sigma V^\top$$

- $\mathbb{R}^{m\times n}_{s} \cap \mathcal{H}$ is a smooth manifold with

$$\begin{aligned} \mathrm{T}_X\left(\mathbb{R}^{m\times n}_{s} \cap \mathcal{H}\right) &= \mathrm{T}_X\mathbb{R}^{m\times n}_{s} \cap \mathrm{T}_X\mathcal{H} \\ &= \left\{KV^\top + UJV_\perp^\top :\; K \in \mathrm{T}_H\mathcal{H}^s,\; J \in \mathbb{R}^{s\times(n-s)}\right\} \end{aligned}$$

Closed-form characterization

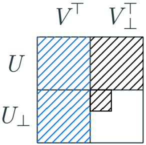
$$\mathrm{T}_X\left(\mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H}\right) = \left\{ \begin{array}{c} KV^\top + UJV_\perp^\top + U_\perp RV_\perp^\top :\; K \in \mathrm{T}_H\mathcal{H}^s \\ J \in \mathbb{R}^{s\times(n-s)},\; R \in \mathbb{R}^{(m-s)\times(n-s)},\; \mathrm{rank}(R) \leq r-s \end{array} \right\}$$

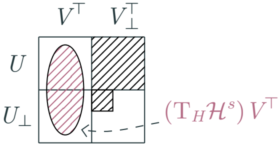
Projection onto the tangent cone

$$\mathcal{P}_{\mathrm{T}_X\left(\mathbb{R}^{m\times n}_{\leq r}\cap\mathcal{H}\right)}(E) = \left(\mathcal{P}_{\mathrm{T}_H\mathcal{H}^s}(EV)\right)V^\top + P_U E P_{V_\perp} + \mathcal{P}_{\mathbb{R}^{m\times n}_{r-s}}\left(P_{U_\perp} E P_{V_\perp}\right)$$

where $P_W := WW^\top$

How orthogonally invariant affects the tangent cone $X = U\Sigma V^\top$





$$\mathrm{T}_X\mathbb{R}^{m\times n}_{\leq r} = \mathbb{R}^{m\times s}\,V^\top + U\mathbb{R}^{s\times(n-s)}\,V_\perp^\top + U_\perp RV_\perp^\top$$

$$\mathrm{T}_X(\mathbb{R}^{m\times n}_{\leq r}\cap\mathcal{H}) = (\mathrm{T}_H\mathcal{H}^s)\,V^\top + U\mathbb{R}^{r\times(n-r)}\,V_\perp^\top + U_\perp RV_\perp^\top$$

Intersection rules for tangent and normal cones

$$\mathrm{T}_X\left(\mathbb{R}^{m\times n}_{\leq r}\cap\mathcal{H}\right) = \mathrm{T}_X\mathbb{R}^{m\times n}_{\leq r}\cap\mathrm{T}_X\mathcal{H}$$

$$\mathrm{N}_X\left(\mathbb{R}^{m\times n}_{\leq r}\cap\mathcal{H}\right) = \mathrm{N}_X\mathbb{R}^{m\times n}_{\leq r}\oplus\mathrm{N}_X\mathcal{H}$$

Definition

A point $X \in \mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H}$ is *stationary* if $-\nabla f(X) \in \mathrm{N}_X(\mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H})$, or equivalently, $\mathcal{P}_{\mathrm{T}_X(\mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H})}(-\nabla f(X)) = 0$

First-order optimality conditions

- Decompose $X = U\Sigma V^\top = HV^\top$ with $\mathrm{rank}(X) = s$ and $H = U\Sigma$
- Projected anti-gradient vanishes

$$\left(\mathcal{P}_{\mathrm{T}_H\mathcal{H}^s}\left(\nabla f(X)\, V\right)\right) V^\top + P_U \nabla f(X) P_{V_\perp} + \mathcal{P}_{\mathbb{R}^{m\times n}_{r-s}}\left(P_{U_\perp} \nabla f(X) P_{V_\perp}\right) = 0$$

- Stationarity at rank-deficient point X with $s < r$

$$-\nabla f(X) \in \left(\mathrm{N}_H \mathcal{H}^s\right) V^\top$$

which reduces to $\nabla f(X) = 0$ when $\mathcal{H} = \mathbb{R}^{m\times n}$

A space-decoupling parameterization

$$\begin{tikzcd}[column sep=small] & \mathcal{M}_h \arrow[dl] \arrow[d] \arrow[rr, hook, "\phi"] & & \mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H} \\ \boxed{\text{rank information?}} & \boxed{h(X) = 0?} & & \end{tikzcd}$$

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A space-decoupling parameterization $\mathcal{M}_h$

- Parameterize the coupled and non-smooth $\mathbb{R}^{m\times n}_{\le r} \cap \mathcal{H}$

$$\mathcal{M}_h := \left\{(X, G) \in \mathbb{R}^{m\times n} \times \mathrm{Gr}(n, n-r) : XG = 0,\ h(X) = 0\right\}$$

where $\mathrm{Gr}(n, n-r) = \{G \in \mathrm{Sym}(n) : G^2 = G, \operatorname{rank}(G) = n - r\}$

Space decoupling - level 3: a smooth parametrization

$$\begin{tikzcd}[column sep=small]
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where $\mathrm{Gr}(n, n-r) = \{G \in \mathrm{Sym}(n) : G^2 = G,\ \mathrm{rank}(G) = n - r\}$

$\mathcal{M}_h$ is a smooth manifold embedded in $\mathbb{R}^{m\times n} \times \mathrm{Sym}(n)$

Represent $(X, G) \in \mathcal{M}_h$ by $X = HV^\top$ and $G = I - VV^\top$ with

$$H \in \mathcal{H}^r \quad \text{and} \quad V \in \mathrm{St}(n, r)$$

Tangent space at $(X, G) \in \mathcal{M}_h$

$$\mathrm{T}_{(X,G)}\mathcal{M}_h = \left\{ \left(KV^\top + HV_p^\top, -V_p V^\top - VV_p^\top \right) : K \in \mathrm{T}_H\mathcal{H}^r,\ V^\top V_p = 0 \right\}$$

Riemannian metric inherited from $\mathcal{E} := \mathbb{R}^{m\times n} \times \mathrm{Sym}(n)$

Represent $(\eta_i, \zeta_i) \in \mathrm{T}_{(X,G)}\mathcal{M}_h$ with $(K_2, V_{p,2})$ $(i = 1, 2)$

$$\langle (\eta_1, \zeta_1), (\eta_2, \zeta_2) \rangle_\omega = \langle K_1, K_2 \rangle + \langle V_{p,1}, V_{p,2} M_{H,\omega} \rangle$$

where $M_{H,\omega} := 2\omega I + H^\top H$

Projection of $(E, Z) \in \mathcal{E}$ onto $\mathrm{T}_{(X,G)}\mathcal{M}_h$

$$\bar{K} = \mathcal{P}_{\mathrm{T}_H\mathcal{H}^r}(EV), \qquad \bar{V}_p = G(E^\top H - 2\omega ZV) M_{H,\omega}^{-1}$$

Represent $(X, G) \in \mathcal{M}_h$ by $X = HV^\top$ and $G = I - VV^\top$ with

$$H \in \mathcal{H}^r \quad \text{and} \quad V \in \mathrm{St}(n, r)$$

Tangent space at $(X, G) \in \mathcal{M}_h$

$$\mathrm{T}_{(X,G)}\mathcal{M}_h = \left\{ \left(KV^\top + HV_p^\top, -V_pV^\top - VV_p^\top \right) : K \in \mathrm{T}_H\mathcal{H}^r,\ V^\top V_p = 0 \right\}$$

Riemannian metric inherited from $\mathcal{E} := \mathbb{R}^{m\times n} \times \mathrm{Sym}(n)$

Represent $(\eta_i, \zeta_i) \in \mathrm{T}_{(X,G)}\mathcal{M}_h$ with $(K_2, V_{p,2})$ $(i = 1, 2)$

$$\langle (\eta_1, \zeta_1), (\eta_2, \zeta_2) \rangle_\omega = \langle K_1, K_2 \rangle + \langle V_{p,1}, V_{p,2} M_{H,\omega} \rangle$$

where $M_{H,\omega} := 2\omega I + H^\top H$

Projection of $(E, Z) \in \mathcal{E}$ onto $\mathrm{T}_{(X,G)}\mathcal{M}_h$

$$\bar{K} = \mathcal{P}_{\mathrm{T}_H\mathcal{H}^r}(EV), \qquad \bar{V}_p = G(E^\top H - 2\omega ZV)M_{H,\omega}^{-1}$$

Complexity: $\mathcal{O}((m+n)r)$

Optimization via the parametrization

$$\begin{tikzcd} \mathcal{M}_h \arrow[d, "\phi"'] \arrow[dr, "\bar{f} := f \circ \phi"] & \\ \mathbb{R}^{m \times n}_{\leq r} \cap \mathcal{H} \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

$$\min_{(X,G) \in \mathcal{M}_h} \bar{f}(X, G)$$

$$\min_{X \in \mathbb{R}^{m \times n}_{\leq r} \cap \mathcal{H}} f(X)$$

Smooth parameterization

- $\phi : \mathcal{M}_h \to \mathbb{R}^{m \times n}_{\leq r} \cap \mathcal{H} : (X, G) \mapsto X$ is a surjection
- Smooth Riemannian optimization problem over $\mathcal{M}_h$
  - Riemannian derivatives
  - Riemannian retractions
  - Vector transports

Riemannian gradient $\nabla_{\mathcal{M}_h}\bar{f}(X, G)$

$$\bar{K} = \mathcal{P}_{\mathrm{T}_H\mathcal{H}^r}\left(\nabla f(X)\,V\right), \qquad \bar{V}_p = G\nabla f(X)^\top H M_{H,\omega}^{-1}.$$

Riemannian Hessian $\nabla^2_{\mathcal{M}_h}\bar{f}(X, G)[\eta, \zeta]$

$$\begin{cases} \bar{K} = \nabla^2_{\mathcal{H}} f(X)[\eta]\,V + \left(I - H M_{H,\omega}^{-1} H^\top\right)\nabla f(X)\,V_p \\ \bar{V}_p = G\left(-V_p[\mathcal{P}_{\mathrm{N}_H\mathcal{H}^r}(\nabla f(X)\,V)]^\top H + (\nabla^2 f(X)[\eta])^\top H + \nabla f(X)^\top W_{H,\omega} K\right) M_{H,\omega}^{-1} \end{cases}$$

First-order retraction

$$\mathrm{R}_{(X,G)}(\eta, \zeta) := (\mathrm{R}_H^{\mathcal{H}^r}(K)(\mathrm{R}_V^{\mathrm{St}}(V_p))^\top,\ I - \mathrm{R}_V^{\mathrm{St}}(V_p)(\mathrm{R}_V^{\mathrm{St}}(V_p))^\top),$$

Second-order retraction

$$\mathrm{R}_{(X,G)}(\eta, \zeta) := ([\mathcal{P}_{\mathcal{H}^r}((X + \eta)\,W)]\,W^\top, I - WW^\top)$$

$$W = L(L^\top L)^{-\frac{1}{2}}, \quad L = V + V_p(I - K^\top H M_{H,\omega}^{-1})$$

Vector transport $\bar{K} = \mathcal{T}_{\tilde{K}}^{\mathcal{H}}(K)$ and $\bar{V}_p = (I - \tilde{V}\tilde{V}^\top)\mathcal{T}_{\tilde{V}_p}^{\mathrm{St}}(V_p)$

Riemannian gradient $\nabla_{\mathcal{M}_h}\bar{f}(X, G)$

$$\bar{K} = \mathcal{P}_{\mathrm{T}_H\mathcal{H}^r}\left(\nabla f(X)\, V\right), \qquad \bar{V}_p = G\nabla f(X)^\top HM_{H,\omega}^{-1}.$$

Riemannian Hessian $\nabla^2_{\mathcal{M}_h}\bar{f}(X, G)[\eta, \zeta]$

$$\begin{cases} \bar{K} = \nabla^2_{\mathcal{H}} f(X)[\eta]\, V + \left(I - HM_{H,\omega}^{-1}H^\top\right)\nabla f(X)\, V_p \\ \bar{V}_p = G\left(-V_p[\mathcal{P}_{\mathrm{N}_H\mathcal{H}^r}(\nabla f(X)\, V)]^\top H + (\nabla^2 f(X)[\eta])^\top H + \nabla f(X)^\top W_{H,\omega}K\right) M_{H,\omega}^{-1} \end{cases}$$

First-order retraction

$$\mathrm{R}_{(X,G)}(\eta, \zeta) := (\mathrm{R}_H^{\mathcal{H}^r}(K)(\mathrm{R}_V^{\mathrm{St}}(V_p))^\top,\ I - \mathrm{R}_V^{\mathrm{St}}(V_p)(\mathrm{R}_V^{\mathrm{St}}(V_p))^\top),$$

Second-order retraction

$$\mathrm{R}_{(X,G)}(\eta, \zeta) := ([\mathcal{P}_{\mathcal{H}^r}((X+\eta)\, W)]\, W^\top, I - WW^\top)$$

$$W = L(L^\top L)^{-\frac{1}{2}}, \quad L = V + V_p(I - K^\top HM_{H,\omega}^{-1})$$

Vector transport $\bar{K} = \mathcal{T}_{\tilde{K}}^{\mathcal{H}}(K)$ and $\bar{V}_p = (I - \tilde{V}\tilde{V}^\top)\mathcal{T}_{\tilde{V}_p}^{\mathrm{St}}(V_p)$

Optimization on $\mathcal{M}_h$ is painless if the geometry of $\mathcal{H}$ is understood

Equivalence of problems

$$\begin{tikzcd}
\mathcal{M}_h \arrow[d, "\phi"'] \arrow[dr, "\bar{f} := f \circ \phi"] & \\
\mathbb{R}^{m \times n}_{\leq r} \cap \mathcal{H} \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

$$\min_{(X,G) \in \mathcal{M}_h} \bar{f}(X, G)$$

$$\min_{X \in \mathbb{R}^{m \times n}_{\leq r} \cap \mathcal{H}} f(X)$$

$$\begin{tikzcd} \mathcal{M}_h \arrow[d, "\phi"'] \arrow[dr, "\bar{f} := f \circ \phi"] & \\ \mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H} \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

$$\min_{(X,G)\in\mathcal{M}_h} \bar{f}(X, G)$$

$$\min_{X\in\mathbb{R}^{m\times n}_{\leq r}\cap\mathcal{H}} f(X)$$

Proposition

The space-decoupling parametrization $\mathcal{M}_h$ of $\mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H}$ satisfies

(i) “$1 \Rightarrow 1$” and “$2 \Rightarrow 2$” hold at (X, G) if and only if $\operatorname{rank}(X) = r$.

(ii) “$2 \Rightarrow 1$” holds everywhere on $\mathcal{M}_h$.

Important ingredients

- $\mathcal{M}_h$ is a complete manifold
- Compact sublevel sets of $f \Longrightarrow$ compact sublevel sets of $\bar{f}$

General convergence theorem

- The sequence $\{(X_k, G_k)\}$ generated by monotone algorithms has at least one accumulation point.
- If the accumulation point (X^*, G^*) is second-order stationary on $\mathcal{M}_h$, then X^* is first-order stationary on $\mathbb{R}^{m \times n}_{\leq r}$.

Specific algorithms

- RGD accumulates to first-order critical points
- RTR accumulates to second-order critical points

Numerical experiments

Approximate $A \in \mathcal{H} = \mathrm{Ob}(m, n)$ with low-rank matrix [Chu et al., 2005 SIAMX]

$$\begin{aligned} \min_{X \in \mathbb{R}^{m \times n}} \quad & \frac{1}{2} \left\| \mathcal{P}_\Omega(X - A) \right\|^2 \\ \mathrm{s.\,t.} \quad & \mathrm{rank}(X) \leq r \\ & \mathrm{diag}(XX^\top) - \mathbf{1} = 0 \end{aligned} \longrightarrow \begin{aligned} \min_{x} \quad & \bar{f}(x) := \frac{1}{2} \left\| \mathcal{P}_\Omega(\phi(x) - A) \right\|_\mathrm{F}^2 \\ \mathrm{s.\,t.} \quad & x \in \mathcal{M}_h \end{aligned}$$

Approximate $A \in \mathcal{H} = \mathrm{Ob}(m, n)$ with low-rank matrix [Chu et al., 2005 SIAMX]

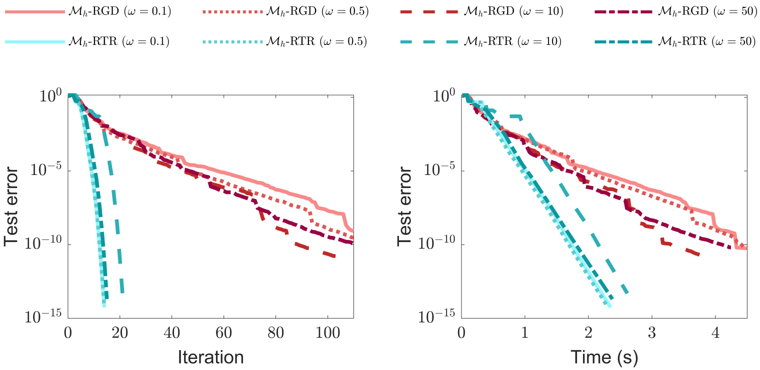
$$\begin{aligned} \min_{X \in \mathbb{R}^{m \times n}} \quad & \frac{1}{2} \|\mathcal{P}_\Omega(X - A)\|^2 \\ \mathrm{s.\,t.} \quad & \mathrm{rank}(X) \leq r \\ & \mathrm{diag}(XX^\top) - \mathbf{1} = 0 \end{aligned} \longrightarrow \begin{aligned} \min_{x} \quad & \bar{f}(x) := \frac{1}{2} \|\mathcal{P}_\Omega(\phi(x) - A)\|_\mathrm{F}^2 \\ \mathrm{s.\,t.} \quad & x \in \mathcal{M}_h \end{aligned}$$

Methods based on Manopt v8.0

- Riemannian gradient descent method (RGD)
- Riemannian trust region method (RTR)

Test with the unbiased rank parameter

- Synthetic $A \in \mathbb{R}^{m\times n}_{\leq r} \cap \mathrm{Ob}(m,n)$, $(m,n) = (5000, 6000)$
  rank parameter $r = r^* = 6$, oversampling factor $\mathrm{OS} = 5$



Test with over-estimated rank parameters

- Synthetic $A \in \mathbb{R}^{m\times n}_{\leq r} \cap \mathrm{Ob}(m,n)$, $(m,n) = (5000, 6000)$
  true rank $r^* = 6$, oversampling factor $\mathrm{OS} = 5$
  over-estimated rank parameters $r = 7, 8, 9, 10 > r^*$

Algorithm	$r=7$		$r=8$		$r=9$		$r=10$	
	Test err.	Time	Test err.	Time	Test err.	Time	Test err.	Time
$\mathcal{M}_h$-RGD	$1.97\text{e}{-11}$	10.23	$1.69\text{e}{-10}$	11.22	$1.49\text{e}{-11}$	9.56	$1.08\text{e}{-9}$	7.30
$\mathcal{M}_h$-RTR	$2.81\text{e}{-15}$	6.03	$3.97\text{e}{-15}$	15.34	$2.41\text{e}{-14}$	4.51	$6.77\text{e}{-14}$	17.08

Measuring problem [Cason et al., 2013 LAA]

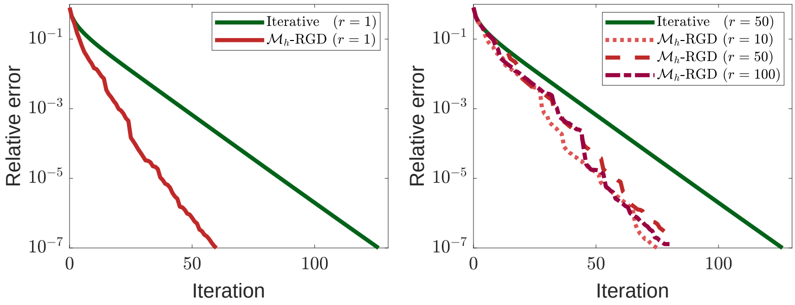
$X \in \mathcal{H} = \mathrm{S_F}(m, n)$ measures the node-to-node similarity between graphs

$$\begin{aligned} \min_{X \in \mathbb{R}^{m\times n}} \quad & -\operatorname{tr}(X^\top \mathcal{L} \circ \mathcal{L}(X)) \\ \text{s.t.} \quad & \operatorname{rank}(X) \le r \\ & \|X\|_\mathrm{F}^2 - 1 = 0 \end{aligned} \longrightarrow \begin{aligned} \min_{x} \quad & -\operatorname{tr}\left(\phi(x)^\top \mathcal{L} \circ \mathcal{L}(\phi(x))\right) \\ \text{s.t.} \quad & x \in \mathcal{M}_h \end{aligned}$$

Test on low-rank solution

Problem setting

- G_A of $m = 2000$ vertices: a single cycle
- G_B of $n = 3000$ vertices: binomial random graph
- Solution of low rank: $r^* = \mathrm{rank}(X^*) = 1$



Left: test with the unbiased rank parameter $r = r^* = 1$

Right: test with over-estimated rank parameters $r = 10, 50, 100 > r^*$

Problem setting

- $m = n$ to generate G_A and G_B: binomial random graphs
- Solution of full rank: $r^* = \mathrm{rank}(X^*) = m$

Numerical results of "$\mathcal{M}_h$-RTR" for different m and r

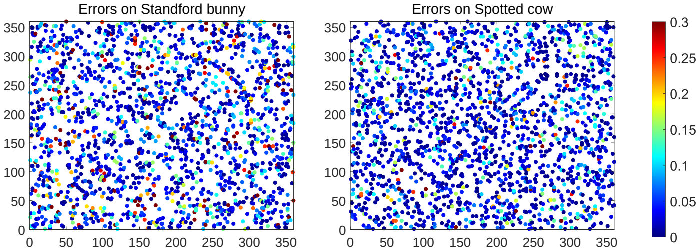
Dimension m	$r = 10\% \times m$		$r = 50\% \times m$		$r = 75\% \times m$		$r = 100\% \times m$	
	Rel. err.	Iter.	Rel. err.	Iter.	Rel. err.	Iter.	Rel. err.	Iter.
200	7.82e−7	38	1.25e−7	31	4.76e−8	26	1.43e−11	4
400	8.23e−7	32	1.85e−7	30	5.92e−8	26	1.07e−11	5
600	7.61e−8	39	5.73e−7	27	3.12e−7	28	8.71e−12	5
800	4.82e−7	37	5.79e−7	28	1.45e−7	21	9.31e−12	5
1000	9.98e−7	37	7.13e−7	31	3.85e−7	28	8.02e−12	5

Synchronization problem

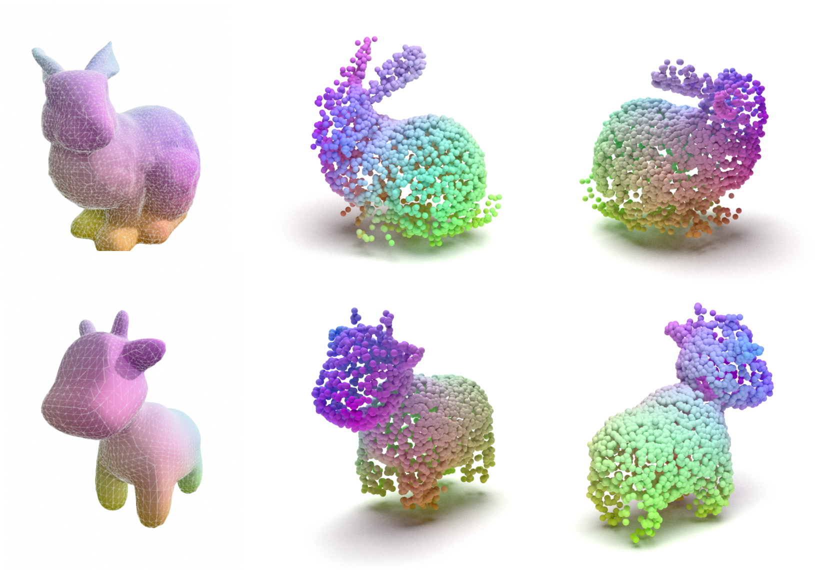
- n cameras with a set of relative rotations $\{\hat{R}_{ij} : (i,j) \in \mathscr{E}\}$
- reconstruct the absolute rotations $\{R_i\}_{i=1}^n$ such that $\hat{R}_{ij} \approx R_j R_i^\top$

$$\begin{aligned} \min_{X\in\mathbb{R}^{m\times n}} \quad & \langle C, XX^\top \rangle \\ \text{s.t.} \quad & \operatorname{rank}(X) \le r \\ & X \in \mathcal{H} = \mathrm{St}(3n,3)^n \end{aligned} \longrightarrow \begin{aligned} \min_{x} \quad & \langle C, \phi(x)\phi(x)^\top \rangle \\ \text{s.t.} \quad & x \in \mathcal{M}_h \end{aligned}$$

Reconstructed errors evaluated by $\|R_i R_j^\top - \hat{R}_{ij}\|_{\mathrm{F}}$



Visualization of the reconstruction



Problem formulation

- State space $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ and Markov model $\mathrm{P} \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{S}|}$
- Hadamard parameterization, $P = X \odot X,\ X \in \mathrm{Ob}(|\mathcal{S}|, |\mathcal{S}|)$

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- Hadamard parameterization, $P = X \odot X$, $X \in \mathrm{Ob}(|\mathcal{S}|, |\mathcal{S}|)$
- Proposed approach

$$\begin{aligned} \min_{X \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{S}|}} \quad & \frac{1}{2} \|X \odot X - \hat{\mathrm{P}}\|_{\mathrm{F}}^2 \\ \mathrm{s.\,t.} \quad & \mathrm{rank}(X) \leq r \\ & \mathrm{diag}(XX^\top) - \mathbf{1} = 0 \end{aligned} \longrightarrow \begin{aligned} \min_{x} \quad & \frac{1}{2} \|\phi(x) \odot \phi(x) - \hat{\mathrm{P}}\|_{\mathrm{F}}^2 \\ \mathrm{s.\,t.} \quad & x \in \mathcal{M}_h \end{aligned}$$

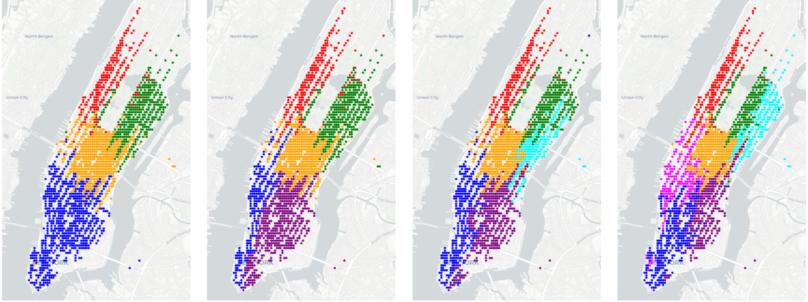
Test on the Manhattan transportation network

Problem setting

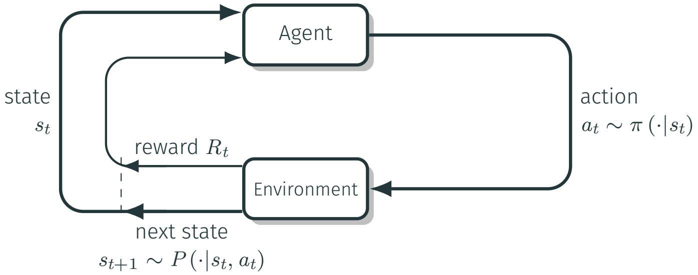
- Dataset of 1.1×10^7 NYC Yellow cab trips
- Model the transportation dynamics as a Markov proces

$$\hat{\mathrm{P}}(i,j) = \frac{\sum_{t=1}^{T} \mathbb{I}\left(s_t^{\text{pickup}} = i,\, s_t^{\text{dropoff}} = j\right)}{\sum_t^T \mathbb{I}\left(s_t^{\text{pickup}} = i\right)}, \quad \text{for} \quad i, j \in \mathcal{S},$$

State compression



$r = 4$ $\qquad$ $r = 5$ $\qquad$ $r = 6$ $\qquad$ $r = 7$



Markov decision process

- State space $\mathcal{S}$, action space $\mathcal{A}$, transition dynamics $\mathrm{P_d} \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|\times|\mathcal{S}|}$
- Reward $R \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}$, policy $\pi \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}$
- Maximize $J(\pi) := \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \mid s_0 \sim \mu, \pi]$

Low-rank strategy

- Hadamard parameterization $\pi(X) := X \odot X,\ X \in \mathrm{Ob}(|\mathcal{S}|, |\mathcal{A}|)$
- Impose the rank constraint $\mathrm{rank}(X) \leq r$

$$\begin{aligned} \min_{X \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{A}|}} \quad & -J(\pi(X)) \\ \mathrm{s.\,t.} \quad & \mathrm{rank}(X) \leq r, \\ & \mathrm{diag}(XX^\top) - \mathbf{1} = 0. \end{aligned} \longrightarrow \begin{aligned} \min_{x} \quad & -J(\pi(\phi(x))) \\ \mathrm{s.\,t.} \quad & x \in \mathcal{M}_h \end{aligned}$$

Low-rank strategy

- Hadamard parameterization $\pi(X) := X \odot X,\ X \in \mathrm{Ob}(|\mathcal{S}|, |\mathcal{A}|)$
- Impose the rank constraint $\mathrm{rank}(X) \leq r$

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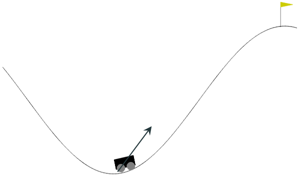
Compared methods

- Low-rank Riemannian policy gradient (LRRPG)
- Q-learning
- REINFORCE

Test in two environments

Pendulum

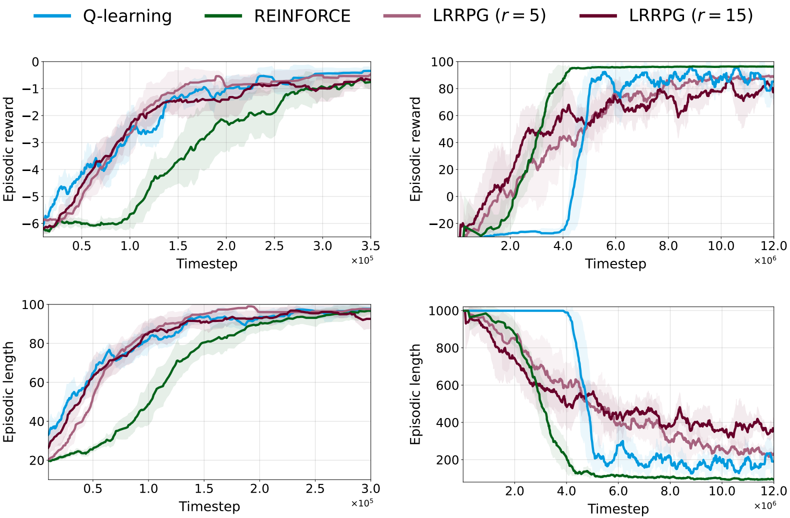
Mountain car



Parameter efficiency

Algorithm	Pendulum			Mountain car		
	Parameters	ρ_{storage}	Reward	Parameters	ρ_{storage}	Reward
REINFORCE	$86,961$	1.00	-0.94	$582,096$	1.00	96.99
Q-learning	$86,961$	1.00	-0.75	$582,096$	1.00	83.95
LRRPG ($r=5$)	$10,810$	0.12	-0.73	$15,485$	0.03	88.31
LRRPG ($r=10$)	$21,620$	0.25	-0.58	$30,970$	0.05	86.44
LRRPG ($r=15$)	$32,430$	0.37	-0.83	$46,455$	0.08	78.23

Convergence curves



Test in “pendulum”

Test in “mountain car”

A space-decoupling framework: four levels

- Relation between rank and orthogonal invariance
- Tangent and normal cones to $\mathbb{R}^{m\times n}_{\leq r} \cap \mathcal{H}$
- Space-decoupling parametrization $\mathcal{M}_h$ and its geometry
- Riemannian optimization on $\mathcal{M}_h$

Future work

- Local convergence results on $\mathcal{M}_h$
- Bounded-rank optimization with additional constraints

Set	Format	First-order	Second-order
$\mathcal{M}$	general	Theorem 1	Theorem 1
$\mathcal{M}_{\leq r}$	matrix	[Schneider-Uschmajew'15 SIOPT]	Proposition 1
$\mathcal{M}^{\mathrm{ht}}_{\leq \mathbf{r}}$	hierarchical Tucker	Proposition 2	Proposition 2
$\mathcal{M}^{\mathrm{tc}}_{\leq \mathbf{r}}$	Tucker	[Gao-Peng-Yuan'25 MP]	Proposition 2
$\mathcal{M}^{\mathrm{tt}}_{\leq \mathbf{r}}$	tensor train	[Kutschan'18 LAA]	Proposition 2
$\mathcal{S}_{\leq r}(n)$	matrix	[Li-Xiu-Zhou'20 JOTA]	Proposition 3
$\mathcal{S}^{+}_{\leq r}(n)$	matrix	[Levin-Kileel-Boumal'25 MP]	Proposition 4
Intersection of sets	**Structured set**	**First-order**	**Second-order**
$\mathcal{M} \cap \mathcal{K}$	general $(\mathcal{M}, \mathcal{K})$	Theorem 2	Theorem 2
$\mathcal{M}_{\leq r} \cap \mathcal{H}$	$\mathcal{H}$ is an affine manifold	[Li-Luo'23 SIOPT]	Appendix C.1
$\mathcal{M}_{\leq r} \cap \mathcal{H}$	$\mathcal{H}$ is orthogonally invariant	[Yang-Gao-Yuan'25]	Appendix C.2
$\mathcal{M}_{\leq r} \cap \mathcal{H}$	$\mathcal{H}$ is hyperbolic	Appendix C.3	Appendix C.3
$\mathcal{S}_{\leq r}(n) \cap \mathcal{U}$	$\mathcal{U} = \{X \mid \|X\|_{\mathrm{F}}^2 = 1\}$	Appendix D.1	Appendix D.1
$\mathcal{S}^{+}_{\leq r}(n) \cap \mathcal{U}$	$\mathcal{U} = \{X \mid \mathcal{A}(X) = b\}$	[Levin-Kileel-Boumal'25 MP]	Appendix D.2

Conclusion and perspectives

A space-decoupling framework: four levels

- Relation between rank and orthogonal invariance
- Tangent and normal cones to $\mathbb{R}^{m\times n}_{\le r} \cap \mathcal{H}$
- Space-decoupling parametrization $\mathcal{M}_h$ and its geometry
- Riemannian optimization on $\mathcal{M}_h$

Future work

- Local convergence results on $\mathcal{M}_h$
- Bounded-rank optimization with additional constraints

References

- **Yan Yang**, Bin Gao, Ya-xiang Yuan. *A space-decoupling framework for optimization on bounded-rank matrices with orthogonally invariant constraints.* Under major revisions at Mathematical Programming
- **Yan Yang**, Bin Gao, Ya-xiang Yuan. *Variational analysis of determinantal varieties.*

- Code is publicly available from https://github.com/UCAS-YanYang

Thanks for your attention!

Email: yangyan@amss.ac.cn